On the Economics of Global Warming with Threshold Effects

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Abstract

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I develop a simplified model designed to study optimal emission policies in the face of a discontinuous and irreversible jump in economic costs caused by an abrupt shift in climatic conditions. I examine the infinite-horizon discounted dynamic programming problem where the choice variable is greenhouse gas (GHG) emissions. Emissions are necessary for economic welfare, but lead to an accumulation of atmospheric GHG. If atmospheric GHG surpasses a threshold, the costs of global warming jump discontinuously. The value function and stationary optimal policy are characterized. It is shown that the optimal policy and path are non-monotone, with reduced emissions as the level of GHG approaches the threshold and an increase in emissions just before the threshold is reached. The value function and optimal policy are comprised of segments, with each segment corresponding to a finite number of periods in which it is optimal to reach the threshold. The optimal emission is discontinuous in the initial state.

I calibrate the model to a particular global warming threshold, that of a potential collapse in the North Atlantic thermohaline circulation process. Because the results of long-horizon cost benefit studies are almost entirely driven by the choice of discount factor, I study the interaction of discount factor uncertainty and this particular long-horizon threshold. Accounting for discount factor uncertainty facilitates recognition of an associated real option value. The option value arises because ex ante preference uncertainty and irreversibility create a potential value to delaying the time at which a central planner would optimally choose to surpass the irreversible threshold. I use the calibrated threshold model to compute one measure of the magnitude of this option value. I show that for distributions calibrated from the empirical literature, the impact of discount factor uncertainty is significantly larger than the impact of catastrophic cost uncertainty. With any given discount factor, the present value of uncertainty in the catastrophic costs is virtually negligible. With any given catastrophic cost, the present value of uncertainty in the discount factor is quite significant.

Key words:

global warming, threshold, dynamic programming, exhaustible resources, stochastic discount factor, gamma discounting.

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For M. & T.

Chapter 1

Introduction

According to the Third Assessment Report of the Intergovernmental Panel on Climate Change (IPCC) [25], "the globally averaged surface temperature is projected to increase by 1.4-5.8 degrees Celsius (2.5-10.4 degrees Fahrenheit) over the period 1990 to 2100," in the absence of policy measures designed to reduce the likelihood of such warming. According to The Guardian newspaper, a draft of the next IPCC report due to be circulated in April 2006, "will say that scientists are unable to place a reliable upper limit on how quickly the atmosphere will warm as carbon dioxide levels increase."[1] Global warming is caused by the increase in atmospheric levels of greenhouse gases (GHGs), primarily carbon dioxide[2]. Carbon dioxide (CO_2) is released by the generation of energy from fossil fuels. Current efforts to avert global warming are primarily concerned with CO_2 emissions cuts. Any such cuts will impose a significant economic cost. Since most emissions come from fossil fuel energy generation, cuts are likely to involve reduced energy consumption, leading to lower output of goods and services. Costs will be incurred if economies are required to invest in more expensive but less carbon-intensive technologies to generate energy. Hence, there is a need to

[1] Guardian Weekly, March 3-9, 2006, page 9.

[2] The Third Assessment Report states, "For the SRES illustrative scenarios, relative to the year 2000, the global mean radiative forcing due to GHGs continues to increase through the 21st century, with the fraction due to CO_2 projected to increase from slightly more than half to about three-quarters." pp.12, Synthesis Report, Summary for Policymakers.

evaluate the future benefit of averting global warming against the current economic costs.

Several well-known empirical models, referred to as integrated assessment models (IAMs) have been developed to carry out cost-benefit analysis of global warming brought about by GHG emissions.[3] Most of these studies have focused on the steady and ongoing accumulation of atmospheric GHGs, a gradual increase in mean global temperatures, and a consequent gradual increase in the costs of global warming. Given the enormity of the problem at hand, there is still much to be worked out even if the costs of global warming do turn out to increase gradually. However, the dominant approach of studying the economics of gradual climate change ignores the potential implications of abrupt shifts in climatic conditions.

I develop a simplified model designed to study optimal emission policies in the face of a discontinuous and irreversible jump in economic costs caused by an abrupt shift in climatic conditions. I examine the infinite-horizon discounted dynamic programming problem where the choice variable is greenhouse gas (GHG) emissions. Emissions are necessary for economic welfare, but lead to an accumulation of atmospheric GHG. If atmospheric GHG surpasses a threshold, the costs of global warming jump discontinuously. The value function and stationary optimal policy are characterized. It is shown that the optimal policy and path are non-monotone, with reduced emissions as the level of GHG approaches the threshold and an increase in emissions just before the threshold is reached. The value function and optimal policy are comprised of segments, with each segment corresponding to a finite number of periods in which it is optimal to reach the threshold. The optimal emission is discontinuous in the initial state.

I calibrate the model to a particular global warming threshold, that of a potential collapse in the North Atlantic thermohaline circulation (THC) process. Because the results of long-horizon cost benefit studies are almost entirely driven by the choice

[3] For surveys of IAMs, see Weyant (1999) and Springer (2003).

of discount factor, I study the interaction of discount factor uncertainty and this particular long-horizon threshold. Accounting for discount factor uncertainty facilitates recognition of the associated real option value[4]. The option value arises because ex ante preference uncertainty and irreversibility create a potential value to delaying the time at which a central planner would optimally choose to surpass the irreversible threshold. Current stakeholders or future generations, who might choose discount factors higher than the one chosen by the modeler, would place a higher value on averting the threshold than the one computed by the model. Accounting for this potentially higher value facilitates option-adjusted pricing of the natural asset. Without an approach incorporating discount factor uncertainty, no option values of this type are recognized in the cost-benefit analysis and hence assets with very long lives and many potential users in the distant future (such as the climate system) are undervalued relative to assets with far shorter lives and potential users only in the relatively near future (such as consumption goods or capital goods). I use the calibrated threshold model to compute one measure of the magnitude of undervaluation for this option value.[5]

I refer to the magnitude of the discontinuity in costs as the "the catastrophic cost" associated with the threshold. I examine the effect of uncertainty regarding the catastrophic costs associated with a collapse in the North Atlantic THC. I show that for distributions calibrated from the empirical literature, the impact of discount factor uncertainty is significantly larger than the impact of catastrophic cost uncertainty.

[4]There is an extensive literature on the option value or "quasi-option value" associated with irreversibilities and preference uncertainty. For a list of references, see Chichilnisky, Heal and Vercelli (1998).

[5]I note that term "option value" used herein may differ in one important respect from the concept of "option value" used in the environmental literature. The uncertainty surrounding the discount factor modeled here is never resolved: the modeler must choose an optimal policy knowing that the actual realization of the discount factor will not be known in finite time. "Option values" in the literature are usually resolved in finite time. I use the term "option value" to refer to the difference between the value of the associated discounted dynamic programming problem with a constant average discount factor and the weighted average value of the same problem across the distribution of discount factors. Dan O'Flaherty has suggested that this value should be labeled the value from Pascal's Dilemma rather than an "option value."

With any given discount factor, the present value of uncertainty in the catastrophic costs is virtually negligible. With any given catastrophic cost, the present value of uncertainty in the discount factor is quite significant. With constant discount rates of $2\% - 4\%$, catastrophic costs amounting to $5\% - 10\%$ of GDP are virtually irrelevant. With a constant discount rate method at these rates, there appears to be little point in resolving uncertainty about economic damage estimates from abrupt climate change. However, if there is some probability that discount rates ought to be below 2%, then catastrophic costs significantly reduce the optimal initial emission. Given the lack of current consensus on the appropriate discount rate and the inability to determine rates that would be chosen by future generations, any cost-benefit analysis that does not explicitly incorporate discount factor uncertainty is not robust. This is the first analysis to compare the impact of uncertainty in the catastrophic cost versus the impact of uncertainty in the present valuation of that cost and to show that the latter is far more important than the former.

The lack of robustness of the optimal policy under a constant discount factor provides an alternate justification for the precautionary principle. The precautionary principle is usually taken to mean that lack of scientific certainty should not preclude taking actions at reasonable cost to prevent irreversible environmental damage. This is the interpretation cited by Gollier, Jullien & Treich (2000). Discount factor uncertainty is a form of scientific uncertainty: it is a recognition that we do not know what prices ought to be used to translate consumption today into consumption in 200 years. Recognizing that uncertainty would have the planner take an average of optimal policies across the distribution of discount rates which is necessarily more conservative than taking an optimal policy based on a constant average discount factor.

In addition, a constant discount factor cost-benefit analysis leads to a second-order distortion: it distorts the value of information which would resolve catastrophic cost uncertainty. In the calibrated threshold model, at a constant discount rate of 4%, we ought to be willing to pay just 0.04% of our wealth to switch from a world where post-threshold catastrophic costs amount to a sure 10% of GDP to one where they amount to a sure 0%. On the other hand, using an approach incorporating a commonly cited discount factor distribution, we ought to be willing to pay 1.43% of our wealth to make the same switch, an amount that is 36 times as much as the constant discount factor case. This multiple is increasing in the variance of the discount factor distribution.

The remainder of this chapter provides an overview of the issues posed by threshold effects, discount factor uncertainty and the computational demands of solving models which incorporate both. Chapter 2 examines a very simple global warming threshold problem with a general benefit function and a continuous piecewise linear cost function with a single kink. This problem is not motivated by the economics of abrupt climate change, since a discontinuity of the cost or benefit function is an essential feature of a switch in climate states. However, Chapter 2 serves to illustrate an efficient method to construct the optimal solution in a problem with a threshold.

Chapter 3 introduces a discontinuity in the cost function, facilitating analysis of a catastrophic cost from abrupt climate change. The discontinuity leads to a non-monotone optimal policy and emissions path, and complicates the structure of the solution. Chapter 3 lays out the computational algorithm tailored to speed the search for candidate optimal paths that is used in both Chapter 3 and Chapter 4.

Chapter 4 examines the problem of discount factor uncertainty by calibrating a version of the discontinuous threshold problem to the the potential THC collapse and computing solutions for a distribution of discount factors. The results cited earlier are all enumerated in Chapter 4.

1.1 Threshold Effects

There is significant concern on the part of many oceanographers and earth scientists, that the accumulation of atmospheric GHG may push the global climate to a tipping point, leading to abrupt and discontinuous climate changes. Several climate models predict threshold effects in the process of climate change, due to switches between multiple steady states (Broecker et al. (1985), Bryan (1986), Manabe & Stouffer (1988) and Stocker & Wright (1991)). Among potential thresholds cited by climate scientists is the possibility of abrupt climate change induced by a collapse in the North Atlantic thermohaline circulation. See Gagosian (2003) for an introductory description and NRC Committee on Abrupt Climate Change (2002) for a more detailed analysis. There is geological evidence that this type of posited shutdown has occurred in the past, inducing mini Ice Ages in western Europe about 12,700 years and 8,200 years ago (see Rahmstorf 1995). Other potential climate thresholds are the potential collapse of the Antarctic ice sheet (see Oppenheimer 1998) or a positive feedback effect on warming from the release of methane buried in melting permafrost.

Abrupt climate change events are deemed "unlikely" by the Third Assessment Report, meaning that according the panel of expert opinion, they have a 1-10% probability of occurring. Such events are nevertheless worthy of extensive research attention since the consequences may be catastrophic and irreversible. This dissertation is concerned with discontinuous changes in the economic costs of global warming at a threshold level of atmospheric GHG. The analysis is based on a discontinuity in cost, not in temperature. Given the possibility of switches among equilibria, it is likely that discontinuities in economic costs would be observed even if global average temperature changes were continuous.

A number of purely analytical papers study threshold effects: Farzin (1996), Clarke & Reed (1994), Tsur & Zemel (1996). In addition, there are a handful of purely numerical models devoted to thresholds: Gjerde et al. (1999) and Keller, Bolker and Bradford (2004). Threshold models incorporating parameter uncertainty can lead to computational intractability, making it difficult to study both parameter uncertainty and thresholds . High discount factors pose special computational problems, as outlined in the next section. The focus of my contribution is to meld analytical solutions and tailored numerical optimization methods to avoid the curse of dimensionality that besets long-horizon, high discount factor dynamic programming problems. The low computational cost of finding the optimum in this approach facilitates an examination of model risk because many perturbations are feasible. As a result, I am able to focus on the interaction between the discount factor uncertainty and a specific global warming threshold.

1.2 A Remark on Computation

Dynamic global warming models which involve purely numerical optimization are forced to ignore the infinite horizon nature of the problem, because computers cannot solve infinite horizon problems. The justification for instituting a near-term finite horizon (200-500 years) is that at "reasonable" discount rates (often 3%), the future stops mattering after 200 years or so. An alternative justification is that it is quite speculative to make extrapolations beyond 200 years or so. This type of argument is appropriate for quite short horizons in the context of problems in finance, but it is inappropriate in an environmental problem such as this. Firstly, by applying a relatively high constant discount rate of 3%, we ensure that the distant future is irrelevant. Secondly, in the context of irreversibilities, it is more speculative to assume that the discount rates used by humans two centuries hence will be 3% than it would be to explicitly model discount factor uncertainty. To deal with lower discount rates, the finite horizon in a purely numerical model would have to expand to 2500 years or more. The reason the models are generally limited to 500 years or so is due to computational complexity. The reality is that few existing numerical models can actually be extended out to 2500 years given the computational costs.

Discount factors close to 1 ensure that the present value of utilities in the far future are non-negligible. Consequently when the optimization routine checks candidate paths for optimality, it must check paths that have many more periods than in the case with lower discount factors. It is this requirement which slows down the search for optimal paths when the problem has a high discount factor. As the number of periods increases, the curse of dimensionality ensures that improved hardware resources foreseeable in the near future will not on their own allow us to solve the problem. The addition of a threshold further complicates the computational problem since it is no longer sufficient to examine marginal benefits and costs only. I implement a computationally efficient method that fully utilizes the analytical results of chapters 2 and 3 to reduce the computation speed while ensuring a global optimum. Unlike numerical threshold models, I can guarantee that the solution is an ex-ante global optimum. The thrust of my approach is to streamline the search for the optimal path by melding analytical solutions and bespoke numerical methods and then use the liberated computing resources to analyze parameter uncertainty and thresholds.

1.3 Discount Factor Uncertainty

There is a significant heterogeneity of views on the current and future discount rates that are appropriate for long horizon environmental cost-benefit problems. There has been an ongoing debate on the appropriate value since at least Ramsey (1928).[6] An approach often used in careful cost-benefit analysis is to equate the discount rate to the sum of a) the pure rate of time preference and b) the product of the elasticity of marginal public utility and the growth rate of wealth. For ethical reasons, term a) the pure rate of time preference, is often taken to be close to zero: Arrow et al (2004) suggest a range of 0-0.5%. Term b) is more difficult to establish because the growth rate of future wealth is quite uncertain, and in the context of long horizon environmental problems with irreversible catastrophes could be negative or only slightly positive. The analysis of Arrow et al. suggests that even in the recent past, the growth rate of a measure of wealth that includes natural capital may be

[6] A range of different philosophical approaches are outlined in Portney and Weyant (1999).

insignificantly different from zero. Hence, the appropriate net discount rate may well be significantly lower than 2%, with the discount factor being correspondingly higher than 0.98.

Despite some consensus on the approach of computing an appropriate discount rate, there remains considerable disagreement on the actual rate that ought to be used, presumably because there is disagreement on the appropriate values that the parameters should take. In a survey asking 2,160 economists their best guess of the appropriate discount rate, Weitzman (2001) received a range of responses that appear to fit a gamma-distribution with a mean rate of 4% (or an approximate discount factor of 0.96) and a standard deviation of 3%.

I follow the approach of directly incorporating uncertainty about discount factors in a very simple way, as suggested by Weitzman. In particular, I calibrate the discontinuous threshold problem of solving for the optimal emissions path given a constant discount factor δ. I then ask how the optimal emission path differs for different constant discount factors. I assume that the potential distribution of discount rates corresponding to the discount factors is a gamma distribution as estimated by Weitzman's survey. Finally, I ask how a planner, who chooses to implement a policy that is a weighted average of the optimal policies for each constant discount factor, would behave. The weights used in the weighted average are those implied by the parametrized gamma distribution. This approach is rational for a planner who does not expect to be able to influence the discount factor distribution and does not expect to learn anything about the distribution until it is too late to change his policy. I show that a combination of discount factor uncertainty and irreversibility leads to a material option value to delaying the date at which to surpass the threshold. I give an estimate of the sensitivity of this option value to a mean-preserving spread in the gamma distribution.

Newell & Pizer have examined numerically the impact of stochastic discount factors in a dynamic model of climate change without thresholds. I am specifically interested here in the interaction between the stochastic discount factor and the threshold and its implications for option value. This paper bolsters the importance of discount factor uncertainty demonstrated by Newell & Pizer by showing that in the absence of a stochastic discount factor approach, threshold effects are virtually irrelevant in cost-benefit analysis.

In a numerical model with a finite horizon, Gjerde et al. (1999) carry out an analysis of optimal climate policy under the possibility of a future general catastrophe. My results are qualitatively consistent with theirs but are conceptually and quantitatively different. Although all their main results are stated for a discount rate of 3%, Gjerde et al. carry out sensitivity analysis on the discount rate. They do not however use a distribution for this parameter. My results are consistent with their sensitivity analysis insofar as optimal policies become dramatically more conservative at lower discount rates. They use a distribution for catastrophic damage and so are able to report an estimate of the value of the resolution of catastrophic damage uncertainty but they do not report a corresponding value for the resolution of discount factor uncertainty because they do not use a discount factor distribution. Because this paper treats catastrophic damage uncertainty and discount factor uncertainty in a similar way, my study demonstrates that the expected value of the resolution of discount factor uncertainty is far higher than the expected value of the resolution of catastrophic damage uncertainty. Gjerde et al. also restrict their planning horizon to 240 years. It is possible that as a result, the negative impact of a catastrophe even with a low discount rate, is understated.

Keller, Bolker and Bradford (2004) use a numerical model of an uncertain climate threshold that is also calibrated to a THC collapse. However, they use a fixed discount rate of 3%. Their model is notable because their time profile of optimal emissions, computed numerically, is the same as the one derived analytically in Chapter 3[7].

[7] When an earlier version of chapter 3 was presented at the Princeton Environmental Institute in January 2001, the late David Bradford pointed out the similarity between Figure 3-1 in Chapter 3 and Figure 5 of Keller, Bolker and Bradford (2004).

1.4 Concluding Remarks

The model developed here incorporates many levels of aggregation and simplification. Other than the discontinuity, the cost function and the climate dynamics are as simple as they can be. Uncertainty is introduced in a very limited and specific manner: the threshold itself is assumed to be known, but its future cost impact and the present value of that cost impact are modeled as unknown. An alternative way to model uncertainty would be to allow the threshold to be unknown, with a possible known future cost impact. Throughout, I have assumed that all emissions are irreversible, whereas a more realistic assumption would be that they are only partially reversible. I have ignored strategic aspects of global warming, assuming that a central planner can implement the optimal policy.

Despite these weaknesses, I hope that the approach I have used illuminates some important aspects of the interaction between thresholds and discount factor uncertainty. In particular, I make an attempt at measuring the magnitude of option value embedded in the current climate system. In future research I hope to compare the magnitudes of the positive option value embedded in a long-horizon natural asset with the negative option value embedded in the sunk costs of emission-reduction technologies. It has been argued that because these two option values work in opposite directions, the net impact of accounting for optionality in environmental policy is unclear[8]. A proper accounting of horizons and underlying variance in these two options is necessary to sign the net impact. I believe the hybrid analytical-computational approach used in this paper can facilitate an examination of this question in an infinite horizon framework with multiple sources of uncertainty. The magnitude of option values are driven by the length of the option and the variance of the underlying random variable. It is my conjecture that with an infinite horizon model that fully calibrates the lives and variances of the natural asset and the emission-reducing capital good,

[8] See, for example, Pindyck (2000).

the option value embedded in the capital good will be far smaller than the option value embedded in the natural asset[9].

[9] An analogue to this problem exists in option-adjusted bond pricing. The error in pricing short-term bonds without option-adjustment is far smaller than the error in pricing an otherwise equivalent long-term bond.

Chapter 2

A Simple Continuous Cost Threshold

This chapter considers a very simple global warming threshold problem with a general benefit function and a continuous piecewise linear cost function with a single kink. The continuity of the cost function prevents its application to problems of abrupt climate change, where a discontinuity is crucial. However, it serves to illustrate an efficient method to construct the optimal solution in a problem with a threshold. A related model has been studied in the continuous time framework by Farzin (1996). Farzin does not attempt to construct an analytical solution outside linearized steady states, using simulation to search for the solution in those cases. Here I construct the analytical solution and fully characterize the policy function, value function and the optimal emissions path in a discrete time framework. The results of this chapter do not differ qualitatively from Farzin's analysis and the contribution is merely one of exhaustively specifying the structure of the value function throughout the domain of the state variable. However, it is a necessary step in the distillation of the computational procedures required to solve the more complicated problem in chapters 3 and 4. By using a discrete time method, I explicitly construct the operations that are subsequently necessary to construct the optimal solution efficiently using a computational algorithm. I construct an operator which serves as a basis for an algorithm to find the optimum in the problem with a continuous cost function as in this chapter as well as with a discontinuous cost function as in Chapters 3 and 4. The reason I substitute simulation for direct construction at this stage is so that I can limit the leakage of computational resources in the simplest aspects of the optimization problem.

I show that continuity of the cost function combined with an Inada condition ensures that the threshold will optimally be surpassed in finite time. The optimal emission path is constructed using a backward induction approach: since the threshold is optimally surpassed in finite time, I solve for the value function at the threshold; I then enumerate all optimal methods of arriving at the threshold. The policy and value functions are continuous and non-increasing in the state, with segments whose domains are readily computed from the primitives.

2.1 The Model

Time is modeled discretely. The global emission of (a scalar index of) greenhouse gases during period t is denoted a_t. The stock of atmospheric greenhouse gases at the beginning of period t is denoted g_t. The law of motion for total GHG is

$$g_{t+1} = g_t + a_t \tag{2.1}$$

$$g \geqslant 0, a \geqslant 0$$

The objective function of the problem is given by

$$\sum_{t=0}^{\infty} \delta^t \left(h[a_t] - c[g_t] \right) \tag{2.2}$$

$$0 < \delta < 1$$

where δ denotes the global discount factor. The benefit function h represents the relationship between the global output of goods and services, and emissions of GHG. Output depends on emissions through energy usage. Thus h determines the economic cost of reduced emissions in the form of lost GDP. The benefit function is assumed to be twice continuously differentiable and to satisfy:

Condition 2.1 Strict Monotonicity and Concavity

$$h' > 0, h'' < 0$$

Condition 2.2 Inada Condition

$$\lim_{a \to 0} h'(a) = \infty \tag{2.3}$$

The marginal cost of global warming depends on whether a threshold level γ of GHG has been surpassed:

$$c[g_t] = \begin{cases} \alpha_1 + c_1 g_t & \text{if } g_t \leqslant \gamma \\ \alpha_2 + c_2 g_t & \text{if } g_t \geqslant \gamma \end{cases}$$

We assume continuity of the cost function at γ. Hence, we can replace α_2.

$$\begin{aligned} \alpha_1 + c_1\gamma &= \alpha_2 + c_2\gamma \\ &\implies \alpha_2 = \alpha_1 - \gamma(c_2 - c_1) \end{aligned}$$

Hence, we have the following reduced form cost function:

$$c[g_t] = \begin{cases} \alpha_1 + c_1 g_t & \text{if } g_t \leqslant \gamma \\ \alpha_1 - \gamma(c_2 - c_1) + c_2 g_t & \text{if } g_t \geqslant \gamma \end{cases} \tag{2.4}$$

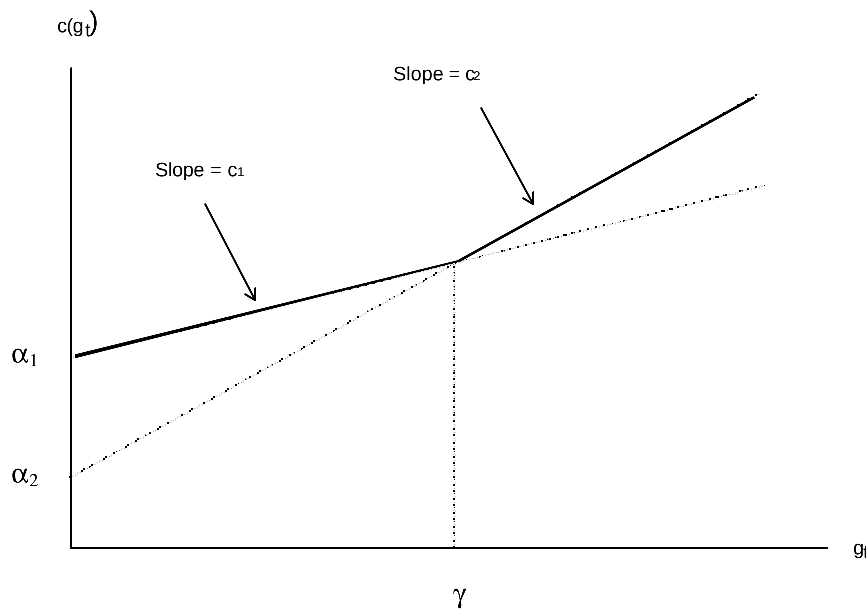

Figure 2-1: Continuous Cost Function with Single Threshold

We assume further that the marginal cost increases at γ and that $c(g_t) \geqslant 0$ for all $g > 0$.

Condition 2.3 Positive and Increasing Cost Function

$$\begin{aligned} \alpha_1 &\geqslant 0 \\ 0 &\leqslant c_1 < c_2 \end{aligned}$$

Figure 2-1 displays a representative cost function.

This model is a 1-country version of the model presented by Dutta and Radner [14], with three modifications:

i) the cost function $c[g_t]$ is piecewise linear, whereas it is assumed to be linear in [14];

ii) the law of motion of GHG in this model (2.1) assumes that a unit of GHG, once emitted, remains in the atmosphere for ever. In [14], some portion of atmospheric GHG may be absorbed in each period due to natural processes. Hence, the law of motion in [14] is $g_{t+1} = \sigma g_t + a_t$, where $\sigma \in [0, 1]$ is the absorption parameter. We assume that $\sigma = 1$. This assumption is not necessary for the results herein to hold, but it simplifies the analysis considerably. What is required is that σ be sufficiently close to 1 such that under an optimal path of emissions, g_t is non-decreasing. In other words, the optimal a_t needs to be greater than $(1 - \sigma)g_t$ always.

iii) Dutta and Radner's benefit function h need not satisfy an Inada condition of the form in (2.3). This condition is necessary in the present model to ensure existence of an optimal strategy involving non-zero emissions. In addition, the benefit function in Dutta and Radner attains a maximum at a finite level of emissions. This is not a necessary condition for this analysis[1], and in the present paper the opposite is assumed: h is strictly increasing.

2.2 Construction of the Optimal Policy Function

Remark 2.1 *If $g_t > \gamma$, then $\widehat{a}_2$ is the constant optimal emission, where $h'(\widehat{a}_2) = \frac{\delta c_2}{1-\delta}$. The value function is linear in g_t :*

$$V(g_t) = \frac{1}{1-\delta}\left(h(\widehat{a}_2) - \frac{\delta c_2 \widehat{a}_2}{1-\delta} + \gamma(c_2 - c_1) - \alpha_1 - c_2 g_t\right) \tag{2.5}$$

These results are a special case of a general problem examined in [14].

When $g_t = \gamma$, is it possible that an optimal solution implies that γ will never be surpassed? Can the increase in marginal cost from c_1 to c_2 be sufficiently high such that it is not optimal to surpass γ? An implication of Jensen's Inequality is that this

[1]The existence of a maximum of $h[\cdot]$ in Dutta and Radner assures the existence of a solution, even when the cost of global warming is zero. For any non-zero cost, however, the assumption of a maximum of $h[\cdot]$ is not required for existence.

cannot be so. For it to be inoptimal to emit 0 at γ, the following inequality must hold:

$$\begin{aligned} h(0)-\alpha_1-c_1\gamma &< h(\widehat{a}_2)-\alpha_1-c_1\gamma-\frac{\delta c_2\widehat{a}_2}{1-\delta} \quad (2.6)\\ h(0) &< h(\widehat{a}_2)-\frac{\delta c_2\widehat{a}_2}{1-\delta}\\ \frac{\delta c_2\widehat{a}_2}{1-\delta} &< h(\widehat{a}_2)-h(0) \quad (2.7)\end{aligned}$$

This follows from the concavity of h.

It is worth noting here that whether to stay at γ or to jump over depends not on the difference between c_1 and c_2 but on c_2 alone. Why? Because at γ, c_1 is no longer relevant.

It is easy to show that that even though the horizon is infinite, in the optimal solution, γ will be reached in finite time.

Lemma 2.1 *If $g=\gamma$, then the optimal policy is to emit $\widehat{a}$ forever. If $g \leqslant \gamma$, then under the optimal policy, $g_t \geqslant \gamma$ for some finite t.*

Proof. *Step 1:* Let us call any policy which leads to $g' > \gamma$ a surpassing policy. We show that among surpassing policies, an emission level of $\widehat{a}_2$ is optimal. The payoff to emitting $\widehat{a}_2$ for ever is

$$h(\widehat{a}_2)-\alpha_1-c_1g+\frac{\delta}{1-\delta}(h(\widehat{a}_2)-\frac{\delta c_2}{1-\delta}\widehat{a}_2+\gamma\,(c_2-c_1)-\alpha_1-c_2(g+\widehat{a}_2)) \quad (2.8)$$

Any optimal surpassing policy must involve constant emissions of $\widehat{a}_2$ after period 1, since we know that the optimal policy is $\widehat{a}_2$ when $g>\gamma$. Hence the payoff to emitting any $a > \gamma - g$ in period 1 is given by

$$h(a)-\alpha_1-c_1g+\frac{\delta}{1-\delta}(h(\widehat{a}_2)-\frac{\delta c_2}{1-\delta}\widehat{a}_2+\gamma\,(c_2-c_1)-\alpha_1-c_2(g+a)) \quad (2.9)$$

If we subtract (2.9) from (2.8), we get

$$h(\widehat{a}_2) - \frac{\delta c_2}{1-\delta}\widehat{a}_2 - (h(a) - \frac{\delta c_2}{1-\delta}a) \tag{2.10}$$

Since by definition, $\widehat{a}_2$ maximizes $h(a) - \frac{\delta c_2}{1-\delta}a$, we know that (2.10) is positive. This implies that the payoff to emitting $\widehat{a}_2$ for ever is greater than emitting any other $a > \gamma - g$ in the first period.

Step 2: We show that the optimal policy at γ is $\widehat{a}_2$. At γ, we may either emit 0 or $\widehat{a}_2$ (since $\widehat{a}_2$ is optimal among surpassing policies). From (2.5), we know that the payoff to emitting $\widehat{a}_2$ at γ is a constant v_γ:

$$\begin{aligned} v_\gamma &\equiv h(\widehat{a}_2) - \alpha_1 - c_1\gamma + \delta V(\gamma + \widehat{a}_2) \\ &= h(\widehat{a}_2) - \alpha_1 - c_1\gamma \\ &\quad + \frac{\delta}{1-\delta}\left(h(\widehat{a}_2) - \frac{\delta c_2 \widehat{a}_2}{1-\delta} + \gamma(c_2 - c_1) - \alpha_1 - c_2\gamma - c_2\widehat{a}_2\right) \\ &= \frac{1}{1-\delta}\left(h(\widehat{a}_2) - \alpha_1 - c_1\gamma - \frac{\delta c_2 \widehat{a}_2}{1-\delta}\right) \end{aligned} \tag{2.11}$$

Emitting 0 for T periods followed by $\widehat{a}_2$ provides a payoff which is less than v_γ by (2.7). Hence, if $g = \gamma$, the optimal policy is $\widehat{a}_2$ and the value is the constant v_γ.[2]

Step 3: I now prove the Lemma by contradiction. Suppose that $g_t < \gamma$ for all t under an optimal policy $\{a_t^*\}$. Then $\{a_t^*\}$ must be optimal for a g arbitrarily close to γ. At such a g, we always have the option of emitting $(\gamma - g)$ followed by $\widehat{a}_2$, with a consequent payoff of

$$h(\gamma - g) - \alpha_1 - c_1 g + \delta v_\gamma$$

[2] In contrast to Chapter 3, continuity of the cost function is sufficient to ensure that γ is reached in finite time. (In Chapter 3, discontinuity of the cost function implies that we will need an inequality of levels embodied in Condition 3.3).

On the other hand, under the posited optimal policy, the payoff is

$$h(a_0^*) - \alpha_1 - c_1 g + \delta v$$

where v is the continuation payoff under $\{a_t^*\}$. We need to show that for g sufficiently close to γ,

$$h(a_0^*) + \delta v < h(\gamma - g) + \delta v_\gamma$$

By re-arranging, we obtain

$$\delta(v - v_\gamma) < h(\gamma - g) - h(a_0^*) \tag{2.12}$$

Since $a_0^* < \gamma - g$, the RHS of (2.12) is always positive. As $g \to \gamma$, the continuation payoff from $\{a_t^*\}$ must approach $\frac{h(0)-\alpha_1-c_1\gamma}{1-\delta}$. Hence, as $g \to \gamma$, the LHS of (2.12) approaches

$$\begin{aligned} &\delta(\frac{h(0) - \alpha_1 - c_1\gamma}{1-\delta} - \frac{h(\widehat{a}_2) - \alpha_1 - c_1\gamma - \frac{\delta c_2 \widehat{a}_2}{1-\delta}}{1-\delta}) \\ = &\frac{\delta}{1-\delta}\left(h(0) - h(\widehat{a}_2) + \frac{\delta c_2 \widehat{a}_2}{1-\delta}\right) \end{aligned}$$

By (2.7), this quantity is negative. Hence as $g \to \gamma$, the LHS of (2.12) becomes negative, proving the inequality. Thus, the policy $\{a_t^*\}$ cannot be optimal. ∎

Since γ is reached in finite time under the optimal policy, we can break up the problem into two parts. Firstly, for any given g, we can determine $T^*[g]$, the optimal number of periods in which to reach γ. Secondly, we can determine the optimal path of emissions before we reach γ. We examine the latter question first: for any given T, we derive the optimal path of emissions $\{a_t^*\}_{t=0}^{T-1}$ and deduce its properties from the principles of dynamic programming.

Definition 2.1 *Let* $T^*[g_0] \equiv \min[t \mid g_t \geqslant \gamma]$ *when* g_t *is induced by the optimal policy* $\{a_t^*\}$.

Proposition 2.1 describes a useful characteristic of the optimal path. The optimal path of emissions in periods prior to T^* is determined by the intertemporal relationship embodied in an Euler equation.

Proposition 2.1 *Suppose* $g_0 < \gamma$. *Then*

i) $\forall t \geqslant T^*, a_t^* = \widehat{a}_2$

ii) $a_{T^*-1}^* = \widehat{a}_2$ *or* $\gamma - g_{T^*-1}$

iii) $\forall t, 0 \leqslant t \leqslant T^* - 2,\ a_t^* \in (0, \gamma - g_t)$. *Moreover,* $\{a_t^*\}_{t=0}^{T^*-2}$ *satisfies the Euler equation*

$$h'[a_t^*] = \delta h'[a_{t+1}^*] + \delta c_1 \tag{2.13}$$

Finally, $\{a_t^*\}_{t=0}^{T^*-1}$ *is decreasing in* t.

Proof. *Step 1:* We know that $g_{T^*} \geqslant \gamma$. If $g_{T^*} = \gamma$, then from Lemma 2.1, we have i). If $g_{T^*} > \gamma$, then from Theorem 1 of Dutta and Radner [14], we have i).

Step 2: If $g_{T^*} > \gamma$, then from the definition of T^*, a_{T^*-1} was a surpassing emission. We know from the proof of Lemma 2.1 that the optimal surpassing emission is $\widehat{a}_2$. If $g_{T^*} = \gamma$, then clearly, $a_{T^*-1} = \gamma - g_{T^*-1}$. Hence, ii) is shown.

Step 3: We show that in any two adjacent periods t and $t+1$, when both g_t and g_{t+1} are less than γ, $a_t^* \in (0, \gamma - g_t)$. Pick any $\gamma - g > 0$. Let $\theta^* = \gamma - g - a^*$ be the optimal next period remaining emissions. We know that θ^* must solve the following two-period maximization problem:

$$\max_{\theta \in [0, \gamma - g]} h(\gamma - g - \theta) + \delta \left(h(\theta - \theta') - c_1(\gamma - g - \theta) \right)$$

If it were not true that the solution to the above lies in $(0, \gamma - g)$, then we must have that $\theta = 0$ or $\theta = \gamma - g$.

Case 1: $(\theta = 0)$

The FOC are:

$$\delta h'(\theta - \theta') \leqslant h'(\gamma - g - \theta) - \delta c_1$$

If $\theta = 0$, then $\theta' = 0$, and the FOC reduces to

$$\infty = \delta h'(0) \leqslant h'(\gamma - g) - \delta c_1 < h'(0) = \infty$$

Case 2: $(\theta = \gamma - g)$

The FOC are:

$$\delta h'(\gamma - g - \theta') \geqslant h'(0) - \delta c_1$$

This is not possible unless $\theta' = \gamma - g$, and by induction $\theta'' = \gamma - g$ *ad infinitum.* The only way this situation can arise is if on the entire path we have zero emissions. But this is evidently suboptimal by Condition 2.1.

Step 4: Since the optimal solution is interior, i.e. $a_t^* \in (0, \gamma - g_t)$, the optimal emission path must satisfy the Euler equation. For $g < \gamma$, the Bellman equation for this problem is

$$V(g) = \max_{g' \in [g, \gamma]} \{h(g' - g) - c_1 g + \delta V(g')\}$$

The first-order and envelope conditions for the problem are

$$\begin{aligned} h'(g' - g) &= -\delta V'(g') \\ V'(g) &= -h'(g' - g) - c_1 \end{aligned}$$

which yield the Euler equation

$$h'(a_t^*) = \delta h'(a_{t+1}^*) + \delta c_1 \tag{2.14}$$

Since h is concave, $a_{t+1}^* < a_t^*$, which implies that $\{a_t^*\}_{t=0}^{T^*-2}$ is decreasing in t. ■

The Euler equation allows us to construct by backward induction the emissions along an optimal path, and the optimal policy function. It will be useful to develop a system of notation for emissions along a path that meets (2.13). We define the function $L^i(a_t)$ for $i = 0, 1, ...$ as the function that returns an emission i periods before a_t such that adjacent period emissions satisfy (2.13).

Definition 2.2 *Let* $L^1 : \mathbb{R} \to \mathbb{R}$ *s.t.* $h'(L^1(a_t)) = \delta h'(a_t) + \delta c_1$

Definition 2.3 *Let* $L^i : \mathbb{R} \to \mathbb{R}$ *for* $i = 0, 2, 3, 4, ...$ *s.t.* $L^0(a_t) = a_t$, *and* $L^2(a_t) = L^1(L^1(a_t))$, $L^3(a_t) = L^1(L^1(L^1(a_t)))$ ...

Furthermore, we define some critical values of g. We let $\overline{\gamma}_i$ denote the values of g at which we may start an i-period emission sequence that satisfies (2.13), such that γ is reached exactly with an emission of $\widehat{a}_2$.

Definition 2.4 *Let* $\overline{\gamma}_i = \gamma - \sum_{j=0}^{i-1} L^j(\widehat{a}_2)$ *for* $i = 1, 2, 3, ...$

We let $\underline{\gamma}_i$ denote the values of g at which we may start an i-period emission sequence that satisfies (2.13), such that γ is reached exactly with an emission of $L(\widehat{a}_2)$.

Definition 2.5 *Let* $\underline{\gamma}_i = \gamma - \sum_{j=1}^{i} L^j(\widehat{a}_2)$ *for* $i = 1, 2, 3, ...$

Figure 2-2 illustrates the relative placement of $\overline{\gamma}_i$ and $\underline{\gamma}_i$.

Remark 2.2 *By construction, the width of each interval* $\left[\overline{\gamma}_{i+1}, \underline{\gamma}_i\right]$ *is the constant* $\widehat{a}_2$.

Similarly, the width of each interval $\left[\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right]$ *is* $L^{i+1}(\widehat{a}_2) - \widehat{a}_2$.

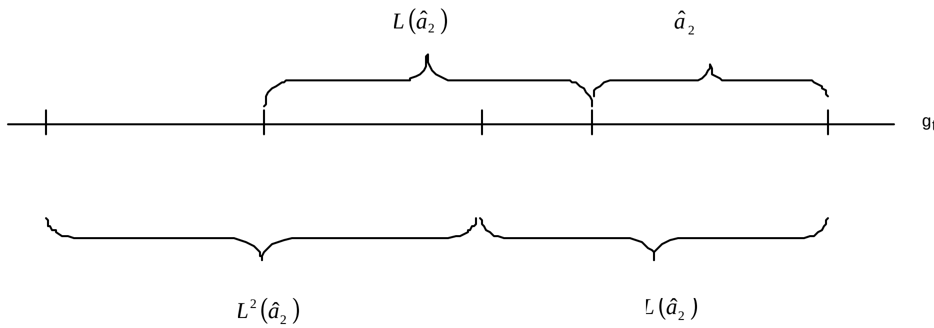

Figure 2-2: Placement of $\overline{\gamma}_i$ and $\underline{\gamma}_i$

We show that in the region $[\overline{\gamma}_1, \gamma]$, the optimal policy is to emit $\widehat{a}_2$.

Lemma 2.2 *If $g \in [\overline{\gamma}_1, \gamma]$, then the optimal policy is to emit $\widehat{a}_2$ forever.*

Proof. We know from the proof of Lemma 2.1 that $\widehat{a}_2$ is optimal among surpassing policies. We need only show that non-surpassing policies are not optimal in the interval $[\gamma - \widehat{a}_2, \gamma]$. We first consider a non-surpassing policy that emits $(\gamma - g)$ and compare its payoff to emitting $\widehat{a}_2$. The payoff to emitting $(\gamma - g)$ is:

$$h(\gamma - g) - \alpha_1 - c_1 g_t + \frac{\delta}{1-\delta}\left(h(\widehat{a}_2) - \alpha_1 - c_1\gamma - \frac{\delta c_2 \widehat{a}_2}{1-\delta}\right)$$

The payoff to emitting $\widehat{a}_2$ is given by:

$$h(\widehat{a}_2) - \alpha_1 - c_1 g_t + \frac{\delta}{1-\delta}(h(\widehat{a}_2) - \frac{\delta c_2}{1-\delta}\widehat{a}_2 + \gamma\,(c_2 - c_1) - \alpha_1 - c_2(g_t + \widehat{a}_2))$$

We need to show that

$$\begin{aligned} & h(\widehat{a}_2) + \frac{\delta}{1-\delta}(h(\widehat{a}_2) - \frac{\delta c_2}{1-\delta}\widehat{a}_2 + \gamma\,(c_2 - c_1) - \alpha_1 - c_2(g_t + \widehat{a}_2)) \\ > \;& h(\gamma - g) + \frac{\delta}{1-\delta}\left(h(\widehat{a}_2) - \alpha_1 - c_1\gamma - \frac{\delta c_2 \widehat{a}_2}{1-\delta}\right) \end{aligned}$$

or

$$\begin{aligned} h(\widehat{a}_2) + \frac{\delta}{1-\delta}(c_2\gamma - c_2(g_t + \widehat{a}_2)) &> h(\gamma - g_t) \\ h(\widehat{a}_2) - \frac{\delta c_2 \widehat{a}_2}{1-\delta} &> h(\gamma - g_t) - \frac{\delta c_2}{1-\delta}(\gamma - g_t) \end{aligned}$$

which is true since $\widehat{a}_2$ maximizes $h(a) - \frac{\delta c_2 a}{1-\delta}$.

What about non-surpassing policies which involve emissions less than $(\gamma - g)$? From Proposition 2.1, such policies must display a decreasing path of emissions, with the last emission before γ is surpassed being $\widehat{a}_2$ or $\gamma - g$. Clearly that is not feasible in $[\gamma - \widehat{a}_2, \gamma]$. ■

An implication of (2.13) is that any optimal emission that leads to a $g' \in (\overline{\gamma}_1, \gamma)$ must be an emission of $L^1(\widehat{a}_2)$. Since $L^1(\widehat{a}_2) > \widehat{a}_2$, there is a region of g to the left of $\overline{\gamma}_1$, where an emission of $L^1(\widehat{a}_2)$ would cause a $g' > \gamma$. In that region, we now show that it is optimal to emit $\gamma - g$, rather than $L^1(\widehat{a}_2)$.

Lemma 2.3 *If $g \in [\underline{\gamma}_1, \overline{\gamma}_1]$, then the optimal policy is to emit $\gamma - g$.*

Proof. *An emission of $\widehat{a}_2$ cannot be optimal, since it would lead to a $g' \in [\overline{\gamma}_1, \gamma]$, whence the optimal emission will be $\widehat{a}_2$. Two adjacent periods with emissions of $\widehat{a}_2$ before γ is reached does not satisfy (2.13). By definition, $L^1(\widehat{a}_2)$ would be a surpassing emission in this region, and we have already shown that among surpassing emissions, $\widehat{a}_2$ is optimal. Hence, the optimal emission in this region is non-surpassing. Since the optimal emission is non-surpassing, we can choose from emitting $\gamma - g$ or start a sequence of emissions that satisfy (2.13) and postpone the surpassing emission to a later period. Such a sequence must involve an eventual surpassing emission less than $\widehat{a}_2$, which we know to be inoptimal.* ■

Having constructed the optimal policy for $g \geqslant \underline{\gamma}_1$, we can generate the optimal policy below $\underline{\gamma}_1$ by backward induction. We first need one more definition.

Definition 2.6 *Let* $\widetilde{a}_{T,t} = L^{T-t}(a)$*, s.t.* $\sum_{s=0}^{T-1} L^s(a) = \gamma - g$*, for* $T = 1, 2, 3, ...$ *and* $t \leqslant T$*.*

Hence, $\widetilde{a}_{T,t}$ denotes the tth emission in an emission sequence that emits $\gamma - g$ over T periods, where adjacent period emissions satisfy the Euler equation rule.

Proposition 2.2 shows that the optimal policy function is comprised of segments. In certain regions of g, the optimal policy function is flat. In such regions, the size of the constant optimal emission is determined by the number of periods before γ is surpassed, and is related to $\widehat{a}_2$, which is the constant level of optimal emission when γ is to be surpassed in 1 period. In the remaining regions of g, the optimal policy function is given by $\widetilde{a}_{T,1}$, for some T.

Proposition 2.2 *For* $g \in \left[\overline{\gamma}_{i+1}, \underline{\gamma}_i\right]$ *for* $i = 1, 2, ...$ *, the optimal policy is to emit* $L^i(\widehat{a}_2)$*. For* $g \in \left[\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right]$ *for* $i = 1, 2, ...$ *, the optimal policy is to emit* $\widetilde{a}_{i+1,1}$*.*

Proof. Step 1: *For* $g \in \left[\overline{\gamma}_{i+1}, \underline{\gamma}_i\right]$*, a policy of emitting* $L^i(\widehat{a}_2)$ *leads in to* $g' \in \left[\overline{\gamma}_i, \underline{\gamma}_{i-1}\right]$*, by definition. Under the proposed policy, we would then emit* $L^{i-1}(\widehat{a}_2)$*. Eventually, in* i *periods, such a policy will lead to a* $g \in [\overline{\gamma}_1, \gamma]$*, whereupon we would emit* $\widehat{a}_2$*. By construction, we will have satisfied the Euler equation condition for all emissions till we surpass* γ*. The marginal cost of a unit of emission is determined by the number of periods an optimal policy will lead us to sojourn with* $g \leqslant \gamma$*. In particular, since we will sojourn with* $g \leqslant \gamma$ *for* i *periods under this policy (excluding period* 0*), the marginal cost a unit of emissions in period* 0 *is given by a finite sum of periods when we incur* δc_1 *and an infinite sum of periods when we incur* δc_2*:*

$$\textit{Marginal Cost of emissions in period } 0 = \frac{\left(1 - \delta^i\right)\delta c_1 + \delta^{i+1} c_2}{1 - \delta} \tag{2.15}$$

We show that the marginal benefit of a unit of emissions is equated to (2.15) at $L^i(\widehat{a}_2)$*. By definition,* $h'(L^1(\widehat{a}_2)) = \delta h'(\widehat{a}_2) + \delta c_1$*. Similarly,* $h'(L^i(\widehat{a}_2)) = \delta h'(L^{i-1}(\widehat{a}_2)) + \delta c_1$*.*

By iterated substitution, we get

$$\begin{aligned} h'(L^i(\widehat{a}_2)) &= \delta h'(L^{i-1}(\widehat{a}_2)) + \delta c_1 \\ &= \delta^i h'(\widehat{a}_2) + \delta c_1 \left(\frac{1-\delta^i}{1-\delta}\right) \\ &= \frac{\delta^{i+1} c_2}{1-\delta} + \delta c_1 \left(\frac{1-\delta^i}{1-\delta}\right) \end{aligned}$$

Hence, such a plan is optimal.

Step 2: *Let* $g \in \left[\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right]$. *At* $\underline{\gamma}_{i+1}$, *we know that* $L^{i+1}(\widehat{a}_2)$ *is optimal, and* γ *is reached exactly in* $i+1$ *periods (by construction). Such a plan causes us to surpass* γ *in* $i+2$ *periods. Suppose at any* $g \in \left(\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right)$, *we employ a similar plan of emissions* $(L^{i+1}(\widehat{a}_2), L^i(\widehat{a}_2), L^{i-1}(\widehat{a}_2)...L^0(\widehat{a}_2))$. *This plan would lead us to surpass* γ *in* $i+1$ *periods. Since* $L^{i+1}(\widehat{a}_2)$ *is optimal for surpassing in* $i+2$ *periods, such a plan can no longer be optimal. In particular, since we surpass* γ *in* 1 *less period, the marginal cost of emissions in period* 0 *would be as follows:*

$$\begin{aligned} \textit{MC of emissions in period } 0 &= \frac{\left(1-\delta^i\right)\delta c_1 + \delta^{i+1} c_2}{1-\delta} \\ &> \frac{\left(1-\delta^{i+1}\right)\delta c_1 + \delta^{i+2} c_2}{1-\delta} = h'(L^{i+1}(\widehat{a}_2)) \end{aligned}$$

Hence, by emitting $L^{i+1}(\widehat{a}_2)$ *in* $g \in \left(\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right)$, *the marginal cost of emissions would exceed the marginal benefit, suggesting that we should reduce the initial emission. Suppose instead that we emit* $(L^i(\widehat{a}_2), L^{i-1}(\widehat{a}_2)...L^0(\widehat{a}_2))$, *which we know to be optimal at* $\overline{\gamma}_{i+1}$..*This plan would lead us to surpass* γ *in* $i+2$ *periods. But now the marginal cost of emissions in period* 0 *would be as follows:*

$$\begin{aligned} \textit{MC of emissions in period } 0 &= \frac{\left(1-\delta^{i+1}\right)\delta c_1 + \delta^{i+2} c_2}{1-\delta} \\ &< \frac{\left(1-\delta^i\right)\delta c_1 + \delta^{i+1} c_2}{1-\delta} = h'(L^i(\widehat{a}_2)) \end{aligned}$$

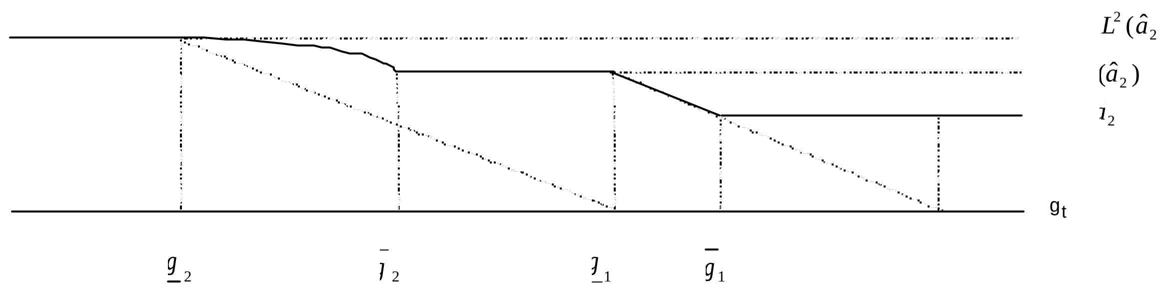

Figure 2-3: Optimal Policy Function

Hence, by emitting $L^i(\widehat{a}_2)$ in $g \in \left(\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right)$, the marginal benefit of emissions would exceed the marginal cost, suggesting that we should increase the initial emission. By construction, $\widetilde{a}_{i+1,1}$ is the emission that lies between $L^i(\widehat{a}_2)$ and $L^{i+1}(\widehat{a}_2)$ such that we reach γ exactly in $i+1$ periods. ■

Figure 2-3 displays the optimal policy function.

We can show that if we start out with a $g_0 \in \left[\overline{\gamma}_{i+1}, \underline{\gamma}_i\right]$, then along the optimal path, there is no $g_t \in \left[\underline{\gamma}_j, \overline{\gamma}_j\right]$. In other words, if the optimal initial period emission is $L^i(\widehat{a}_2)$, it will never be optimal to emit anything other than $L^j(\widehat{a}_2)$ for some $j \leqslant i$. Conversely, if we start out with a $g_0 \in \left[\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right]$, then along the optimal path, there is no $g_t \in \left[\overline{\gamma}_j, \underline{\gamma}_j\right]$.

Remark 2.3 *If the surpassing emission in a T-period plan is $a < \widehat{a}_2$, it cannot be optimal. Similarly, if the surpassing $a > L(\widehat{a}_2)$, then it cannot be optimal.*

2.3 Construction of the Value Function

We now construct the value function $W(g)$, simply by substituting the optimal policy for each g.

$W(g) =$

$$\begin{cases} V(g) & \text{if } g \geqslant \gamma \\ h(\widehat{a}_2) - \alpha_1 - c_1 g + \delta V(g + \widehat{a}_2) & \text{if } \overline{\gamma}_1 \leqslant g \leqslant \gamma \\ h(\gamma - g) - \alpha_1 - c_1 g + \delta V(\gamma) & \text{if } \underline{\gamma}_1 \leqslant g \leqslant \overline{\gamma}_1 \\ \left\{ \begin{array}{l} \sum_{j=0}^{i} \delta^{i-j} \left(h(L^j(\widehat{a}_2)) - \alpha_1 - c_1 g \right) \\ + \sum_{j=1}^{i} \sum_{k=1}^{j} \delta^k c_1 L^j(\widehat{a}_2) + \delta^{i+1} V(g + \sum_{j=0}^{i} L^j(\widehat{a}_2)) \end{array} \right\} & \text{if } \overline{\gamma}_{i+1} \leqslant g < \underline{\gamma}_i \\ \left\{ \begin{array}{l} \sum_{j=0}^{i} \delta^{i-j} \left(h(\widetilde{a}_{j+1,1}(g)) - \alpha_1 - c_1 g \right) \\ + \sum_{j=1}^{i} \sum_{k=1}^{j} \delta^k c_1 \widetilde{a}_{j+1,1}(g) + \delta^{i+1} V(\gamma) \end{array} \right\} & \text{if } \underline{\gamma}_{i+1} \leqslant g < \overline{\gamma}_{i+1} \end{cases}$$

for $i = 1, 2, 3, \ldots$

We examine the derivative of the value function $W^{'}(g)$ in the different regions. It is easy to show that:

$$W^{'}(g) = \begin{cases} -\left(\frac{c_2}{1-\delta}\right) & \text{if } g \geqslant \gamma \\ -\left(c_1 + \frac{\delta c_2}{1-\delta}\right) & \text{if } \overline{\gamma}_1 \leqslant g \leqslant \gamma \\ -\left(h^{'}(\gamma - g) + c_1\right) & \text{if } \underline{\gamma}_1 \leqslant g \leqslant \overline{\gamma}_1 \\ -\left(\left(\frac{1-\delta^{i+1}}{1-\delta}\right) c_1 + \frac{\delta^{i+1} c_2}{1-\delta}\right) & \text{if } \overline{\gamma}_{i+1} \leqslant g < \underline{\gamma}_i \\ \left\{ \begin{array}{l} \sum_{j=0}^{i} \delta^{i-j} \left(h^{'}(\widetilde{a}_{j+1,1}(g)) \widetilde{a}^{'}_{j+1,1}(g) - c_1 \right) \\ \qquad + \sum_{j=1}^{i} \sum_{k=1}^{j} \delta^k c_1 \widetilde{a}^{'}_{j+1,1}(g) \end{array} \right\} & \text{if } \underline{\gamma}_{i+1} \leqslant g < \overline{\gamma}_{i+1} \end{cases}$$

for $i = 1, 2, 3, \ldots$

The slope of the value function in the flat regions is given by a weighted average of c_1 and c_2. The weight of c_1 is the finite sum of discount factors determined by the number of periods during which $g \leqslant \gamma$. The weight of c_2 is the infinite sum of discount factors after γ is surpassed.

We can show that in the downward-sloping regions, $W^{'}(g)$ is bounded above and below. When $g \in \left[\underline{\gamma}_1, \overline{\gamma}_1\right], W^{'}(g) = -\left(h^{'}(\gamma - g) + c_1\right)$. By definition, $(\gamma - g) \in [\widehat{a}_2, L(\widehat{a}_2)]$ in this region. Therefore, $W^{'}(g)$ ranges from a low of $-\left(c_1 + \frac{\delta c_2}{1-\delta}\right)$ to a high of $-\left((1+\delta)\, c_1 + \frac{\delta^2 c_2}{1-\delta}\right)$ in this region. Hence, $W^{'}(g)$ is continuous at the endpoints of $\left[\underline{\gamma}_1, \overline{\gamma}_1\right]$.

Since the only optimal way to arrive at $\left[\underline{\gamma}_1, \overline{\gamma}_1\right]$ is from another downward-sloping

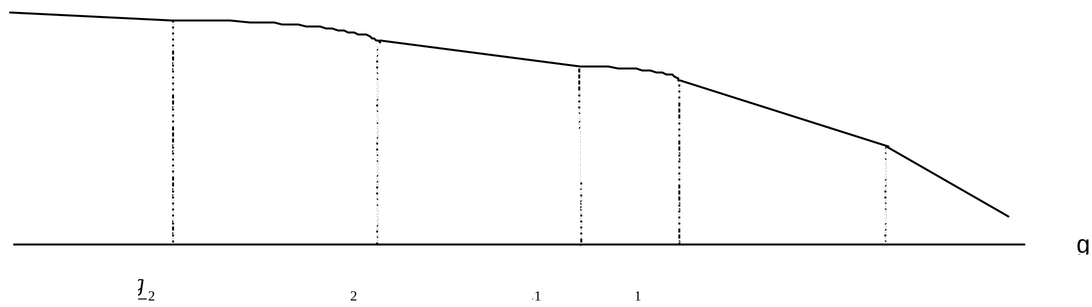

Figure 2-4: Value Function

region, in particular from $\left[\underline{\gamma}_2, \overline{\gamma}_2\right]$, we can bound $W'(g)$ in any downward-sloping region by backward induction. By re-arranging, (2.14) can be expressed thus:

$$W'(g) = \delta W'(g') - c_1$$

where g' is the subsequent period g under the optimal policy. When $g \in \left[\underline{\gamma}_2, \overline{\gamma}_2\right]$, we know that $W'(g') \in \left[-\left(c_1 + \frac{\delta c_2}{1-\delta}\right), -\left((1+\delta)\, c_1 + \frac{\delta^2 c_2}{1-\delta}\right)\right]$. Hence, when $g \in \left[\underline{\gamma}_2, \overline{\gamma}_2\right]$, $W'(g) \in \left[-\left((1+\delta)\, c_1 + \frac{\delta^2 c_2}{1-\delta}\right), -\left(\left(1+\delta+\delta^2\right) c_1 + \frac{\delta^3 c_2}{1-\delta}\right)\right]$. By backward induction, we can bound $W'(g)$ in all downward-sloping regions.

Remark 2.4 *When* $g \in \left[\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right]$,

$$W'(g) \in \left[-\left(\left(\frac{1-\delta^{i+1}}{1-\delta}\right) c_1 + \frac{\delta^{i+1} c_2}{1-\delta}\right), -\left(\left(\frac{1-\delta^{i+2}}{1-\delta}\right) c_1 + \frac{\delta^{i+2} c_2}{1-\delta}\right)\right].$$

Hence, $W'(g)$ is continuous at the endpoints of $\left[\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}\right]$. It follows that $W(g)$ is comprised of a number of downward-sloping linear segments, joined by downward sloping curves which transition smoothly into the linear segments:

2.4 Logarithmic Functional Form

$$h(a) = \ln a$$

Euler Equation:

$$\begin{aligned} \frac{1}{a_t} &= \delta(\frac{1}{a_{t+1}} + c_1) \\ &\Rightarrow a_{t+1} = \delta \frac{a_t}{1 - \delta c_1 a_t} \end{aligned}$$

Remark 2.5 *An implication of the Euler equation in the* ln *benefit case is that sequential emissions are related as follows:*

$$\left\{ a, \frac{\delta a}{1 - \delta c_1 a}, \frac{\delta \left(\frac{\delta a}{1 - \delta c_1 a} \right)}{1 - \delta c_1 \left(\frac{\delta a}{1 - \delta c_1 a} \right)}, \ldots \right\}$$

We can examine the optimal policy in $g \in \left(\underline{\gamma}_2, \overline{\gamma}_2 \right)$. By Proposition 2.2, the optimal policy is to emit $\widetilde{a}_{2,1}$. In the logarithmic benefit function case, $\widetilde{a}_{2,1}$ can be defined by setting the sum of emissions $\widetilde{a}_{2,1}$ and $\widetilde{a}_{2,2}$ equal to $\gamma - g$:

$$\delta \frac{\widetilde{a}_{2,1}}{1 - \delta c_1 \widetilde{a}_{2,1}} + \widetilde{a}_{2,1} = \gamma - g$$

The solution is given by

$$\widetilde{a}_{2,1}(g) = \frac{1}{2\delta c_1} \left(\delta c_1 (\gamma - g) + 1 + \delta - \sqrt{\left(\delta^2 c_1^2 (\gamma - g)^2 - 2\delta c_1 (1 - \delta)(\gamma - g) + 1 + 2\delta + \delta^2 \right)} \right)$$

(Of the two solutions of the quadratic, we have taken the one that involves positive emissions in both periods.)

We now solve for the optimal policy in $g \in \left(\overline{\gamma}_{i+1}, \underline{\gamma}_i\right)$. In the logarithmic benefit function case, $\widehat{a}_1$ and $\widehat{a}_2$ have simple functional forms:

$$\begin{aligned} \widehat{a}_1 &= \frac{1-\delta}{\delta c_1} \\ \widehat{a}_2 &= \frac{1-\delta}{\delta c_2} \end{aligned}$$

Similarly, we can generate the functional form of $L^i(\widehat{a}_2)$:

$$L^i(\widehat{a}_2) = \frac{1-\delta}{\delta^{i+1}(c_2 - c_1) + \delta c_1}$$

Remark 2.6 *We know that $L^i(\widehat{a}_2)$ increases as i increases. How high can $L^i(\widehat{a}_2)$ go?*

$$\lim_{i \to \infty} \frac{1-\delta}{\delta^{i+1}(c_2 - c_1) + \delta c_1} = \frac{1-\delta}{\delta c_1} = \widehat{a}_1$$

This is a sensible bound for $L^i(\widehat{a}_2)$, since it can never be optimal to emit more than $\widehat{a}_1$, however large is $(\gamma - g)$.

2.5 Conclusion

I have derived the structure of the policy and value functions for a simple global warming problem with a continuous threshold. Such a problem cannot be applied to study the economics of abrupt climate change since the costs incurred at the threshold are not discontinuous. However, the method employed to derive the structure of the solution here is used in the next chapter where a discontinuous threshold is analyzed.

Chapter 3

A Discontinuous Threshold

In this chapter, I introduce a discontinuity in the cost function at the threshold γ. I characterize the value function and the stationary optimal policy in the dynamic programming problem faced by a single decision-maker. I show that the optimal policy involves greater reductions in emissions as the level of atmospheric gases approaches the threshold γ. Unlike the case of the continuous threshold studied in Chapter 2, the discontinuous threshold leads to a non-monotone optimal policy and emission path. The optimal emission path involves progressively lower emissions as γ is approached, followed by a discrete jump in emissions just before the threshold is surpassed. As in Chapter 2, the value function and optimal policy are comprised of segments, with each segment corresponding to a finite number of periods in which it is optimal to reach γ. However, in contrast to Chapter 2, the optimal policy is discontinuous in the initial condition.

In this model, the cumulative emissions feasible before the threshold γ is reached may be viewed as an exhaustible resource. Hence, this chapter is closely related to the extensive literature on exhaustible resources and 'cake-eating.'[1] Unlike the focus of that literature, I assume that it is possible to continue emissions, albeit at a cost, even after the resource is exhausted. Effectively, this introduces an outside

[1] See Kemp and Long (1984) for a survey.

option to the standard 'cake-eating' problem with discounting. In this model, the decisionmaker may consume an asymptotically smaller piece of the remaining cake in each period, or she may choose to consume it all and exercise her outside option of emitting beyond the threshold and paying the corresponding cost. The presence of the outside option complicates the optimal policy, creating regions of the state variable where it is optimal to exercise the outside option at different points in the future. The standard 'cake-eating' model (without an outside option) that most closely resembles this paper is developed by Koopmans (1974). In his paper, a minimum level of emission is required for survival in each period, and when the resource is exhausted, the problem effectively ends. Here, the minimum level of emission is endogenous, determined by the relationship between the value of the outside option on the one hand and the value from following a path of asymptotically smaller emissions on the other hand. The value of the outside option is determined by the level of emission that is optimal after γ is reached. This issue does not arise in Koopmans' model because there the problem effectively ends when γ is reached. However, the question of the optimal time to exhaustion arises in both models, and it is here that I benefit from the logic of Koopmans' analysis.

Section 3.1 describes the dynamic programming problem. In Section 3.2, the value function and the optimal policy are characterized. No closed form solution is provided for the optimal policy with a general objective function, but the structure of the policy is derived, relying on a backward induction logic similar to that employed in Chapter 2. Section 3.3 provides a closed-form solution in the case of a logarithmic objective function and carries out comparative statics. Section 3.4 lays out a computational algorithm designed to use all available information to speed up the search for candidate optimal paths. Section 3.5 concludes.

3.1 The Model

Time is modeled discretely. The global emission of (a scalar index of) greenhouse gases during period t is denoted a_t. The stock of atmospheric greenhouse gases at the beginning of period t is denoted g_t. The law of motion for total GHG is

$$g_{t+1} = g_t + a_t \tag{3.1}$$
$$g \geq 0, a \geq 0$$

The objective function of the problem is given by

$$\sum_{t=0}^{\infty} \delta^t \left(h[a_t] - c[g_t]\right) \tag{3.2}$$
$$0 < \delta < 1$$

where δ denotes the global discount factor. The benefit function h represents the relationship between the global output of goods and services, and emissions of GHG. Output depends on emissions through energy usage. Thus h determines the economic cost of reduced emissions in the form of lost GDP. The benefit function is assumed to be twice continuously differentiable and to satisfy:

Condition 3.1 Strict Monotonicity and Concavity

$$h' > 0, h'' < 0 \tag{3.3}$$

Condition 3.2 Inada Condition

$$\lim_{a \to 0} h'(a) = \infty \tag{3.4}$$

The cost of global warming is felt only if a threshold level γ of GHG is surpassed:

$$c[g_t] = \begin{cases} 0 & \text{if } g_t \leqslant \gamma \ {}_{2} \\ \alpha + cg_t & \text{if } g_t > \gamma \end{cases} \tag{3.5}$$
$$\alpha \geq 0, c > 0$$

In other words, before γ is surpassed, there is no immediate benefit from lower levels of atmospheric GHG. The only reason to reduce current emissions is to avert or postpone costs in the (possibly distant) future.

3.2 Optimal Solution

The planner's problem is given by

$$\max_{\{a_t\}} \sum_{t=0}^{\infty} \delta^t \left(h[a_t] - c[g_t]\right) \ \textit{subject to} \tag{3.6}$$
$$g_{t+1} = g_t + a_t$$
$$g_0 \ \textit{given}$$

The natural state variable in this problem is g, and the choice is a.

3.2.1 Solution when $g > \gamma$

We know from Dutta & Radner that if $g > \gamma$, the optimal policy π is to pick a constant level of emission $\widehat{a}$, where $h'(\widehat{a}) = \frac{\delta c}{1-\delta}$. If the threshold level of GHG has already been surpassed, i.e. if $g > \gamma$, then this problem is a special case of that studied by Dutta & Radner. From their work, we know that the optimal policy in that case is to pick a constant level of emission $\widehat{a}$. Such a policy equates the current period marginal benefit of a unit of emission to its lifetime marginal cost. Due to a linear cost function, the marginal cost does not depend on the state variable g. Hence, the optimal policy π does not depend on the state, and is thus a constant.

Let us denote by $V[g]$ the value function to this problem. If $g > \gamma$, then from Theorem 1 of Dutta and Radner [14], we have

$$V[g] = \frac{1}{1-\delta}(h[\widehat{a}] - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - cg) \tag{3.7}$$

Hence, the value is a linear function of g for all $g > \gamma$.

3.2.2 Solution when $g \leqslant \gamma$

We will need to use the optimal policy and value function derived by Dutta & Radner to solve the planner's problem in (3.6) when $g \leqslant \gamma$. The planner's problem here is isomorphic to the well-studied cake-eating problem[3], with a termination value given by (3.7). The analogue of the cake is $\gamma - g$, the total amount of GHG we may emit without surpassing the threshold.

Existence

As shown by Maitra [30], under certain assumptions on the discounted dynamic programming problem, a solution and an optimal strategy can be shown to exist. Unfortunately, the problem in (3.6) fails two of Maitra's conditions:

i) the one-period reward function $h(\cdot) - c(\cdot)$ is not continuous in the state variable at γ.

ii) the set of possible actions, $\mathbb{R}_+$ is not compact.

Hence, when $g \leqslant \gamma$, we cannot guarantee the existence of a solution or a stationary optimal policy. However, we will construct a value function which will be shown to be the solution to (3.6). The properties of this value function will allow us to use part of Maitra's logic to guarantee the existence of a stationary optimal policy π.

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[3] A number of variations of the cake-eating problem with discounting can be found in Kemp and Long (1984).

Solution by Construction

I begin by assuming that the payoff to emitting nothing is sufficiently worse than the cost of surpassing the threshold γ. Condition 3.3 states that the average payoff once γ is reached is better than the average payoff from emitting zero. If this were not so, then the problem would be trivial, since the optimal policy upon reaching the threshold level of GHG would be simply to stop emissions for ever. In that case, the problem would not be qualitatively different from Koopmans (1974), where there is no utility to be derived once γ is reached.

Condition 3.3

$$h(0) < h[\widehat{a}] - \frac{\delta c}{1-\delta}\widehat{a} - \delta(\alpha + c\gamma).$$

An implication of Condition 3.3 is that even though the horizon is infinite, in the optimal solution (if it exists) to this problem, γ will be reached in finite time. I note that this condition was not necessary in Chapter 2, where continuity of the cost function and concavity of the benefit function was sufficient for γ to be optimally surpassed in finite time. The discontinuity of the cost function here necessitates an additional condition in levels.

Lemma 3.1 *If $g = \gamma$, then the optimal policy is to emit $\widehat{a}$ forever. If $g \leqslant \gamma$, then under the optimal policy, $g_t \geq \gamma$ for some finite t.*

Proof. *Step 1:* Let us call any policy which leads to $g' > \gamma$ a *surpassing policy*, (where g' denotes g in the subsequent period). We show that among surpassing policies, an emission level of $\widehat{a}$ is optimal. The payoff to emitting $\widehat{a}$ for ever is

$$h(\widehat{a}) + \frac{\delta}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(g + \widehat{a})) \tag{3.8}$$

Any optimal surpassing policy must involve constant emissions of $\widehat{a}$ after period 1, since we know that the optimal policy is $\widehat{a}$ when $g > \gamma$. Hence the payoff to emitting any $a > \gamma - g$ in period 1 is given by

$$h(a) + \frac{\delta}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(g+a)) \tag{3.9}$$

If we subtract (3.9) from (3.8), we get

$$h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - (h(a) - \frac{\delta c}{1-\delta}a) \tag{3.10}$$

Since by definition, $\widehat{a}$ maximizes $h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a}$, we know that (3.10) is positive. This implies that the payoff to emitting $\widehat{a}$ for ever is greater than emitting any other $a > \gamma - g$ in the first period.

Step 2: We then show that the optimal policy at γ is $\widehat{a}$. At γ, we may either emit 0 or $\widehat{a}$ (since $\widehat{a}$ is optimal among surpassing policies). From (3.7), we know that emitting $\widehat{a}$ provides a constant

$$v_\gamma \equiv \frac{1}{1-\delta}(h[\widehat{a}] - \frac{\delta c}{1-\delta}\widehat{a} - \delta(\alpha + c\gamma)), \tag{3.11}$$

where we have substituted $\delta\alpha$ for α and $\delta c\gamma$ for $c\gamma$ since in this case $(\alpha + c\gamma)$ is borne from period 1 onwards. Emitting 0 for T periods followed by $\widehat{a}$ provides a payoff of $\sum_{t=0}^{T-1} \delta^t h(0) + \delta^T v_\gamma$, which is less than v_γ by Condition 3.3. Hence, if $g = \gamma$, the optimal policy is $\widehat{a}$ and the value is the constant

$$V[\gamma] = v_\gamma \tag{3.12}$$

Step 3: I now prove the Lemma by contradiction. Suppose that $g_t < \gamma$ for all t under an optimal policy $\{a_t^*\}$. Then $\{a_t^*\}$ must be optimal for a g arbitrarily close to γ. At such a g, we always have the option of emitting $(\gamma - g)$ followed by $\widehat{a}$, with a consequent payoff of

$$h(\gamma - g) + \delta v_\gamma$$

On the other hand, under the posited optimal policy, the payoff is

$$h(a_0^*) + \delta v$$

where v is the continuation payoff under $\{a_t^*\}$. We need to show that for g sufficiently close to γ,

$$h(a_0^*) + \delta v < h(\gamma - g) + \delta v_\gamma$$

By re-arranging, we obtain

$$\delta(v - v_\gamma) < h(\gamma - g) - h(a_0^*) \tag{3.13}$$

Since $a_0^* < \gamma - g$, the RHS of (3.13) is always positive. As $g \to \gamma$, the continuation payoff from $\{a_t^*\}$ must approach $\frac{h(0)}{1-\delta}$. But by Condition 3.3,

$$\frac{h(0)}{1-\delta} < v_\gamma$$

Hence as $g \to \gamma$, the LHS of (3.13) becomes negative, proving the inequality. Thus, the policy $\{a_t^*\}$ cannot be optimal. ■

Since γ is reached in finite time under the optimal policy, we can break up the problem into two parts. Firstly, for any given g, we need to determine $T^*[g]$, the optimal number of periods in which to reach γ. Secondly, we need to determine the optimal path of emissions before we reach γ. We examine the latter question first: for any given T, we derive the optimal path of emissions $\{a_t^*\}_{t=0}^{T-1}$ and deduce its properties from the principles of dynamic programming.

Definition 3.1 *Let* $T^*[g_0] \equiv \min[t \mid g_t \geq \gamma]$ *when* g_t *is induced by the optimal policy* $\{a_t^*\}$.

Proposition 2.1 lists some characteristics of the optimal path. In particular, it states that γ may be reached in one of two ways: either it might be surpassed by an emission of $\widehat{a}$, or it may be reached exactly, by an emission of $\gamma - g_{T^*-1}$ in period $T^* - 1$. Furthermore, the optimal path of emissions in periods prior to $T^* - 1$ is determined by the intertemporal relationship embodied in an Euler equation.

Proposition 3.1 *Suppose Condition 3.3 holds and* $g_0 < \gamma$. *Then*

i) $\forall t \geq T^*, a_t^* = \widehat{a}$

ii) $a_{T^*-1}^* = \widehat{a}$ *or* $\gamma - g_{T^*-1}$

iii) $\forall t, 0 \leqslant t \leqslant T^* - 2$, *we have an interior solution, i.e.* $0 < a_t^* < \gamma - g_t$. *Moreover,* $\{a_t^*\}_{t=0}^{T^*-2}$ *satisfies the Euler equation*

$$h'[a_t^*] = \delta h'[a_{t+1}^*]. \tag{3.14}$$

Finally, $\{a_t^*\}_{t=0}^{T^*-1}$ *is decreasing in* t.

Proof. *Step 1:* We know that $g_{T^*} \geq \gamma$. If $g_{T^*} = \gamma$, then from Lemma 2.1, we have i). If $g_{T^*} > \gamma$, then from Theorem 1 of Dutta and Radner [14], we have i).

Step 2: If $g_{T^*} > \gamma$, then from the definition of T^*, a_{T^*-1} was a surpassing emission. We know from the proof of Lemma 2.1 that the optimal surpassing emission is $\widehat{a}$. If $g_{T^*} = \gamma$, then clearly, $a_{T^*-1} = \gamma - g_{T^*-1}$. Hence, ii) is shown.

Step 3: We show that in any two adjacent periods t and $t + 1$, when both g_t and g_{t+1} are less than γ, the optimal solution must be interior. Pick any $\gamma - g > 0$. Let $\theta^* = \gamma - g - a^*$ be the optimal next period remaining emissions. We know that θ^*

must solve the following two-period maximization problem:

$$\max_{\theta\in[0,\gamma-g]} h(\gamma-g-\theta)+\delta h(\theta-\theta^{'})$$

If it were not true that the solution to the above lies in $(0,\gamma-g)$, then we must have that $\theta=0$ or $\theta=\gamma-g$.

Case 1: $(\theta=0)$

The FOC are:

$$\delta h'(\theta-\theta')\leqslant h'(\gamma-g-\theta)$$

If $\theta=0$, then $\theta'=0$, and the FOC reduces to

$$\infty=\delta h'(0)\leqslant h'(\gamma-g)<h'(0)=\infty$$

Case 2: $(\theta=\gamma-g)$

The FOC are:

$$\delta h'(\gamma-g-\theta')\geq h'(0)$$

This is not possible unless $\theta'=\gamma-g$, and by induction $\theta''=\gamma-g$ *ad infinitum*. The only way this situation can arise is if on the entire path we have zero emissions. But this is evidently suboptimal by Condition 3.3.

Step 4: Since the optimal solution is interior, the optimal emission path must satisfy the Euler equation. For $g<\gamma$, the Bellman equation for this problem is

$$V(g)=\max_{a\in[0,\gamma-g]}\{h(a)+\delta V(g+a)\}$$

or, alternatively

$$V(g) = \max_{g' \in [g,\gamma]} \{h(g' - g) + \delta V(g')\}$$

The first-order and envelope conditions for the problem are

$$\begin{aligned} h'(g' - g) &= -\delta V'(g') \\ V'(g) &= -h'(g' - g) \end{aligned}$$

which yield the Euler equation

$$h'(a_t^*) = \delta h'(a_{t+1}^*)$$

Since h is concave, $a_{t+1}^* < a_t^*$, which implies that $\{a_t^*\}_{t=0}^{T^*-2}$ is decreasing in t. ■

A Modified Problem We have shown that under the optimal policy, γ might be reached in one of two ways: it may be surpassed by an emission of $\widehat{a}$, or it may be reached exactly by the emission of $\gamma - g_{T^*-1}$ in period $T^* - 1$. We now examine the optimal path of emissions if γ is reached exactly. We will show later (in Proposition 3.2) that this is the generic optimal path and that a surpassing emission of $\widehat{a}$ occurs under special circumstances. Hence, I examine the original problem with the added constraint that $g_T = \gamma$ for a given T:

Definition 3.2 *For $g_0 < \gamma$, let $\mathrm{V}^T(\mathrm{g})$ for $T = 1, 2, 3, \ldots$, denote the value functions to the problem in (3.6) with the following additional restrictions*

$$\sum_{t=0}^{T-1} a_t = \gamma - g_0 \; for \; a \; given \; T \tag{3.15}$$

$$a_t = \widehat{a}_t \forall t \geq T \tag{3.16}$$

Denote the problem in (3.6) with the above additional constraints for a given T, the

T-constrained *problem.*

Since the Euler equation yields a^*_{t+1} as a function of a^*_t, and the constraint in (3.15) restricts total emissions prior to reaching γ, $\sum_{t=0}^{T-1} a_t$ to $\gamma - g$, we can determine the optimal emission path in a $T-constrained$ problem for a given g and T. Since $h^{'}$ is monotonic, we can invert it and use the Euler equation to obtain

$$\begin{aligned} a^*_t &= f_t[a^*_{t-1}] \; for \; t = 1 \; to \; T-1 \\ a^*_0 &= \gamma - g_0 - \sum_{t=0}^{T-2} f_t[a^*_t] \end{aligned} \tag{3.17}$$

Since $a_t = g_{t+1} - g_t$, the optimal policy is given by

$$a^*_t = f_t[g_t - g_{t-1}]$$

The optimal policy has not yet been shown to be stationary. This is done later, in Proposition 3.5. For a general benefit function h, we cannot solve analytically for the optimal policy π. However, in Section 4, with a restricted functional form, I provide an analytic representation for the stationary optimal policy.

In the next lemma, I examine some properties of the solution to the $T-constrained$ problem. These properties will be used later to characterize the optimal policy in the overall problem in (3.6).

The optimal policy will determine the optimal split of $\gamma - g_0$ into T periods for a given g_0 and T.

Definition 3.3 *Let*

$$\lambda_{T,t} \equiv \frac{a^*_{T,t}}{\gamma - g_0}$$

for $t = 0$ *to* $T-1$.

Then $\lambda_{T,t}$ denotes the optimal fraction of $\gamma - g_0$ that should be emitted in period t, if we are constrained to reaching γ in exactly T periods. Moreover, $\sum_{t=0}^{T-1} \lambda_{T,t} = 1$.

Hence,

$$V^T(g_0) = \sum_{t=0}^{T-1} \delta^t h[\lambda_{T,t}(\gamma - g_0)] + \delta^T v_\gamma \tag{3.18}$$

The concavity of h ensures that for a given t, $\lambda_{T+1,t} < \lambda_{T,t} < \lambda_{T-1,t}$. In other words, if $(\gamma - g_0)$ is to be split into an extra period, then the emissions of the extra period must come from each of the other periods. An implication of this is that the marginal benefit of an extra unit of possible emission is increasing in T. Put differently, the slope of V^T is steeper for higher T.

Lemma 3.2 *For all t, $\lambda_{T+1,t} < \lambda_{T,t}$. In addition, $V^{T\prime}[g_0] > V^{T+1\prime}[g_0]$.*

Proof. See Appendix. ■

The Original Problem Re-examined I have examined the optimal path in a $T - $ *constrained* problem for a given T. I go on to investigate the optimal policy in the original problem in (3.6), rather than the family of $T -$ *constrained* problems. Clearly, if γ is reached exactly, we follow the optimal policy in the $T^* -$ *constrained* problem. However, as stated in Proposition 2.1, it is possible that under the optimal policy, γ is not reached exactly, but is surpassed by an emission of $\widehat{a}$. I show that there exists a left neighborhood of γ, where it is optimal to pick the constant level of emission $\widehat{a}$.

The intuition is as follows: if g is very close to γ, the emissions required by a plan that postpones the surpassing of γ would be very small. In a standard cake-eating problem, this is not significant, since it is always inoptimal to surpass γ. But here, there is an outside option, namely that of surpassing γ and proceeding optimally by emitting $\widehat{a}$ thereafter. If g is very close to γ, the value of surpassing γ and bearing the attendant cost will exceed the value of emitting a small amount and not bearing a cost. Hence, a left neighbourhood of γ will comprise a 'poverty trap,' an area where the state is so bad, that it is optimal not to reduce emissions, but to increase them.

Lemma 3.3 *There exists a* $\gamma_0 \in (\gamma - \widehat{a}, \gamma)$*, such that for* $g \in [\gamma_0, \gamma]$*, the optimal policy is to emit* $\widehat{a}$ *forever.*

Proof. Take a g arbitrarily close to γ, such that $g = \gamma - \epsilon$ for some $\widehat{a} > \epsilon > 0$. Since $\gamma - \widehat{a} < g$, an emission of $\widehat{a}$ will lead to a subsequent period $g' > \gamma$. We have shown (in Step 1 of the proof of Lemma 2.1) that $\widehat{a}$ is optimal among surpassing policies.

We then consider non-surpassing emission plans. The total emissions before γ is reached is ϵ. The value function from emitting ϵ over T^* periods is given by

$$V^{T^*}[\gamma - \epsilon] = \sum_{t=0}^{T^*-1} \delta^t h(\lambda_{T^*,t}\epsilon) + \delta^{T^*} v_\gamma \tag{3.19}$$

As $\epsilon \to 0$, $V^{T^*}[\gamma - \epsilon] \to h(0) + \delta^{T^*} v_\gamma$. From (**??**), the payoff to emitting $\widehat{a}$ in period 0 is

$$h(\widehat{a}) + \frac{\delta}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(\gamma - \epsilon + \widehat{a})) \tag{3.20}$$

which $\to h(\widehat{a}) + \frac{\delta}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(\gamma + \widehat{a})) = v_\gamma$ as $\epsilon \to 0$. By Condition 3.3,

$$\begin{aligned} h(0) &< (1-\delta)v_\gamma \\ h(0) + \delta^{T^*} v_\gamma &< (1 - \delta + \delta^{T^*})v_\gamma \leqslant v_\gamma \end{aligned}$$

This implies that the payoff to emitting $\widehat{a}$ for ever is greater than emitting ϵ over T^* periods in a left neighborhood of γ. The left boundary of the neighborhood is found at γ_0, the value of g where the two payoffs are equal:

$$V^{T^*}[\gamma_0] = h(\widehat{a}) + \frac{\delta}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(\gamma_0 + \widehat{a})) \tag{3.21}$$

We then show that the left boundary $\gamma_0 > \gamma - \widehat{a}$. We have seen that just to the left of γ, the payoff to $\widehat{a}$ beats $V^{T^*}[\gamma - \epsilon]$. I show that at a g just to the right of $\gamma - \widehat{a}$, $V^{T^*}[\gamma - g]$ beats $\widehat{a}$. The marginal loss of reducing emission from $\widehat{a}$ to $\gamma - g$ is $\frac{\delta c}{1-\delta}$. The marginal gain is $\frac{\delta c}{1-\delta} + \delta(\alpha + c\gamma)$, since, the reduced emission postpones by 1 period the onset of the costs $(\alpha + c\gamma)$. Since, (3.20) and $V^{T^*}[\gamma - g]$ are continuous for $g < \gamma$, the point of indifference γ_0 must lie between $\gamma - \widehat{a}$ and γ. ■

I have shown that there exists a left neighborhood of γ where it is optimal to emit $\widehat{a}$. I now show that to the left of this neighborhood, it is always preferable to reach γ exactly in $T^*[g_0]$ periods rather than end up in (γ_0, γ). The intuition is as follows: in period $T^* - 1$, the plan that goes to γ and the plan that goes to (γ_0, γ) both lead to the same stream of costs, starting two periods hence. Therefore, nothing is gained by cutting back in $T^* - 1$ and not going all the way to γ. In addition, since surpassing policies are not optimal to the left of γ_0, it is actually optimal to reach γ exactly in $T^*[g_0]$ periods. This optimal policy has a corresponding value function.

Proposition 3.2 *In the problem in (3.6), for all $g \leqslant \gamma_0$, a policy of reaching γ exactly in $T^*[g_0]$ periods is optimal. For all $g \geq \gamma_0$, the optimal policy is to emit $\widehat{a}$ forever. The value function is given by*

$$V(g_0) = \begin{cases} \sum_{t=0}^{T^*(g_0)-1} \delta^t h[\lambda_{T^*(g_0),t}(\gamma - g_0)] + \delta^{T^*(g_0)} v_\gamma & \text{if } g_0 \leqslant \gamma_0 \\ h(\widehat{a}) + \frac{\delta}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(g_0 + \widehat{a})) & \text{if } \gamma_0 \leqslant g_0 \leqslant \gamma \\ \frac{1}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c g_0) & \text{if } g_0 > \gamma \end{cases} \tag{3.22}$$

Proof. *Step 1*: Consider the emission prescribed in period $T^* - 1$ by the policy of reaching γ exactly in $T^*[g_0]$ periods. Such a policy calls for an emission of $\gamma - g_{T^*-1}$. A consequence of this policy is that we start to bear the cost cg two periods hence. Therefore, $h'[\gamma - g_{T^*-1}] \geq \frac{\delta^2}{1-\delta}c$. If this were not so, we could emit ϵ less, lose $h'[\gamma - g_{T^*-1}]$ and gain $\frac{\delta^2}{1-\delta}c$. Put differently, $h'[z] \geq \frac{\delta^2}{1-\delta}c$ is the FOC to the following problem

$$h[z] + \delta h(\widehat{a}) + \frac{\delta^2}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(\gamma + \widehat{a} - (\gamma - g_{T^*-1} - z)))$$
$$s.t.\ z \leqslant \gamma - g_{T^*-1}$$

Step 2: Since $g_{T^*-1} < \gamma_0$, $\widehat{a}$ is not optimal. Hence, no surpassing policy is optimal. Hence, we need consider only the policy of ending up at (γ_0, γ). The T^* policy calls for an emission of $(\gamma - g_{T^*-1})$, with a consequent payoff

$$\begin{aligned} & h[\gamma - g_{T^*-1}] + \delta v_\gamma \\ = \quad & h[\gamma - g_{T^*-1}] + \delta h(\widehat{a}) + \frac{\delta^2}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(\gamma + \widehat{a})) \end{aligned} \tag{3.23}$$

If instead, we choose to land in (γ_0, γ), then we receive

$$h[\gamma - g_{T^*-1} - x] + \delta h(\widehat{a}) + \frac{\delta^2}{1-\delta}(h(\widehat{a}) - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(\gamma + \widehat{a} - x)) \tag{3.24}$$

where $x \in [0, \gamma - \gamma_0]$ Subtracting (3.24) from (3.23), we get

$$h[\gamma - g_{T^*-1}] - h[\gamma - g_{T^*-1} - x] - \frac{\delta^2}{1-\delta}cx$$

The FOC with respect to x is

$$h'[\gamma - g_{T^*-1} - x] = \frac{\delta^2}{1-\delta}c$$

which implies that $\gamma - g_{T^*-1} - x \geq \gamma - g_{T^*-1}$. Hence the optimal x is 0.

Step 3: The structure of the value function follows directly from the optimal policy. For $g_0 \leqslant \gamma_0$, the value function is the same as the one in the $T^* - constrained$ problem. For $\gamma_0 \leqslant g_0 \leqslant \gamma$, the value function is the return from emitting $\widehat{a}$ once to surpass γ. The value function for $g_0 > \gamma$ derives from (3.7). ■

Since the optimal policy in the $T^* - constrained$ problem is optimal in the original problem when $g < \gamma_0$, and γ_0 is within a one-period emission of γ, the $T^* - constrained$ solution is the generic one. It is only if g_0 lies in the interval (γ_0, γ) that the $T^* - constrained$ policy is not optimal.

Characteristics of the Value Function and Optimal Policy

Proposition 3.2 provides the optimal policy once we find $T^*[g_0]$. We first examine some characteristics of the value function corresponding to the optimal policy. In particular, we can establish that $V[g]$ is continuous everywhere except γ.

We begin by defining values of g which represent points of indifference between the policy of reaching γ in exactly T periods and reaching it in exactly $\widetilde{T} \neq T$ periods.

Definition 3.4 *Let $\widetilde{g}$ be any g s.t.*

$$V^{T^*[g]}[g] = V^T[g] = V^{\widetilde{T}}[g] \text{ for } T \neq \widetilde{T}$$

Lemma 3.4 *$V[g]$ is continuous everywhere except γ.*

Proof. *Step 1*: We know that $V[\gamma] = v_\gamma$, whereas $V[\gamma + \epsilon]$ for arbitrarily small $\epsilon > 0$ is given by

$$V[\gamma + \epsilon] = \frac{1}{1-\delta}(h[\widehat{a}] - \frac{\delta c}{1-\delta}\widehat{a} - \alpha - c(\gamma + \epsilon))$$

the limit of which as $\epsilon \to 0$, is $v_\gamma - \delta(\alpha + c\gamma)$. Hence, V is discontinuous at γ.

Step 2: Clearly, V is continuous with $g > \gamma$. When $g < \gamma$, the points of potential discontinuity are γ_0 and any g where $T^*[g]$ changes. By the continuity of h, V^T is continuous for a given T. Hence, any point where $T^*[g]$ changes is included in the set of $\widetilde{g}'s$. Both γ_0 and any $\widetilde{g}$ are defined by indifference. Hence, V is not discontinuous at such points. ∎

We now apply part of Maitra's logic [30] to show the existence of a stationary optimal policy. We will be able to ignore the discontinuity at γ because Proposition 3.2 shows that the optimal policy at γ is stationary and continuous. We will, however, need to restrict the set of possible actions to a compact subset of $\mathbb{R}_+$. We can do this by picking an upper limit on a which we know will not be exceeded under the optimal policy. One such limit is given by $\overline{a} \equiv \max\{\widehat{a}, \gamma\}$. Hence, we will define the possible action set $A \equiv [0, \overline{a}]$. In addition, we define S as the set of possible states, excluding γ, so that $S \equiv [0, \gamma)$.

Lemma 3.5 *The optimal policy correspondence* π *is stationary.*

Proof. See Appendix. ■

The stationarity of the optimal policy allows us to establish some characteristics of $T^*(g)$. In particular, the next lemma shows that $T^*(g)$ is weakly decreasing in g. In other words, the closer we are to γ, the more swiftly we will choose to reach the threshold. The concavity of h ensures that we gain something from splitting up emissions over more periods. However, due to discounting, there is a limit to such gains [4]. The content of the next lemma is that that limit is reached earlier when g is closer to γ. The stationarity of the optimal policy implies that $T^*(g)$ decreases in steps of 1.

Lemma 3.6 $T^*(g)$ *is weakly decreasing in* g. *Moreover,* $\nexists[g_1, g_2]$ *s.t.* $T^*(g_2) + 1 < T^*(g_1)$ *and* $T^*(g_2) + 1 \notin T^*(g) \forall g \in (g_1, g_2)$. *Put differently,* $T^*(g)$ *decreases in steps of 1.*

Proof. *Step 1*: Suppose, in contradiction to the first result, that $\exists \underline{g} < \overline{g}$ s.t. $T^*(\underline{g}) < T^*(\overline{g})$. By the optimality of $T^*(\overline{g})$, we have

$$V^{T^*(\overline{g})}[\overline{g}] \geq V^{T^*(\underline{g})}[\overline{g}]$$

[4] As Gale (1967) shows, there is no solution to the cake-eating problem without discounting.

We know from Lemma 3.2 that $V^{T^*(\overline{g})\prime}[\cdot] < V^{T^*(\underline{g})\prime}[\cdot]$. Therefore, $V^{T^*(\overline{g})}[\underline{g}] > V^{T^*(\underline{g})}[\underline{g}]$, which contradicts the optimality of $T^*(\underline{g})$.

Step 2: Suppose, in contradiction to the second result, that $\exists[g_1, g_2]$ s.t. $T^*(g_2) + 1 < T^*(g_1)$ and $T^*(g_2) + 1 \notin T^*(g) \forall g \in (g_1, g_2)$. Then there must be an intermediate $\widehat{g}$ s.t. $T^*(g_1) \in T^*(\widehat{g})$ and $T^*(g_2) \in T^*(\widehat{g})$. Hence, at g_2, we are indifferent between a policy of going to γ in $T^*(g_2)$ periods and a policy of going there in $T^*(g_1)$ periods. Suppose we follow $T^*(g_1)$ policy. Then in the subsequent period, $g' > \widehat{g}$ which implies by the first part of this lemma that $T^*(g') \leqslant T^*(g_2)$. Hence, at g', the optimal number of periods in which to reach γ is at most $T^*(g_2)$. But the optimal policy is stationary. This means that in the prior period, the optimal number of periods in which to reach γ is at most $T^*(g_2) + 1$. But then $T^*(g_1)$ cannot have been optimal. ■

An implication of the Lemma 3.6 is that the domain of $T^*(\cdot)$ below γ_0 is partitioned into regions. In the region adjacent to γ_0, $T^*(g) = 1$. In the region to the left of this region, $T^*(g) = 2$, and so on. We can define the boundaries of the regions, which we denote γ_T.

Definition 3.5 *Let γ_T for $T = 1, 2, 3, \ldots$ be the g which solves*

$$\sum_{t=0}^{T-1} \delta^t h(\lambda_{T,t}(\gamma - g)) + \delta^T v_\gamma = \sum_{t=0}^{T} \delta^t h(\lambda_{T+1,t}(\gamma - g)) + \delta^{T+1} v_\gamma.$$

Hence, the γ_T represent the boundaries of each region, where we are indifferent between a policy of going to γ in T periods and a policy of going to γ in $T-1$ periods.

Under the stationary optimal policy, each period's emission must increase g so that g' ends up in the region immediately to the right of the current region. Within each region, T^* is fixed, so π is decreasing in g.

Proposition 3.3 *i) $\exists$ a partition of the interval $[g_0, \gamma_0]$ into a finite number of intervals $[g_0, \gamma_{T-1}]$, $[\gamma_{T-1}, \gamma_{T-2}]$, $\ldots$, $[\gamma_1, \gamma_0]$, s.t. $T^*(g) = T$ if $g \in [\gamma_T, \gamma_{T-1}]$.*

ii) Within any given interval (γ_T, γ_{T-1}), $\pi'(g) < 0$.

iii) If $g \in [\gamma_T, \gamma_{T-1}]$, then $g + \pi(g) \in [\gamma_{T-1}, \gamma_{T-2}]$.

Proof. *i)* We can show by contradiction that $T^*(\gamma_0) = 1$. If $T^*(\gamma_0) > 1$, then $g_1 \in (\gamma_0, \gamma)$ since zero emissions are never optimal. But then, at g_1, the optimal action must be $\widehat{a}$, which means that γ is not reached exactly. Hence, $T^*(\gamma_0)$ cannot be greater than 1. By continuity, $T^*(\gamma_0 - \epsilon) = 1$.

By Lemma 3.6, as g decreases, $T^*(\cdot)$ increases weakly, in steps of 1. By definition, the point of indifference between $T^*(\cdot) = 1$ and $T^*(\cdot) = 2$ is at γ_1. Hence, in the interval $[\gamma_1, \gamma_0], T^*(\cdot) = 1$. The rest follows by induction.

ii) Within any given interval (γ_T, γ_{T-1}), $T^*(\cdot)$ is fixed. Then, for any $\underline{g} < \overline{g}$ s.t. $\underline{g} \in (\gamma_T, \gamma_{T-1})$ and $\overline{g} \in (\gamma_T, \gamma_{T-1})$, $\sum_{t=0}^{T-1} \pi_t(\underline{g}) > \sum_{t=0}^{T-1} \pi_t(\overline{g})$. By concavity of h, $\pi_t(\underline{g}) > \pi_t(\overline{g})$. Hence, $\pi'(g) < 0$.

iii) Suppose not. Then the optimal path after 1 period does not reach γ exactly in $T - 1$ periods. But then, the period 1 emission cannot have been optimal. ■

At the boundaries of the regions, we are indifferent between two future paths to γ, one path taking one less period than the other. The current action under the two paths are however, different. The path with the higher T^* periods will involve lower emissions than the path with the lower T^*. Hence, the optimal action will vary significantly, if there is a small change in the initial state g. Figure 3-1 below illustrates the nature of the optimal policy correspondence:

3.3 Restricted Functional Forms

We now turn to an example with a specific functional form of h. With a general benefit function h, we were unable to characterize the optimal policy correspondence in closed form. When the one-period objective function in a dynamic programming problem such as in (3.6) is of a specific class, then there exist closed-form solutions. Phelps [39] and Hakansson [21] characterize the closed form solutions when the one-period objective function is monotone increasing, strictly concave, with constant relative

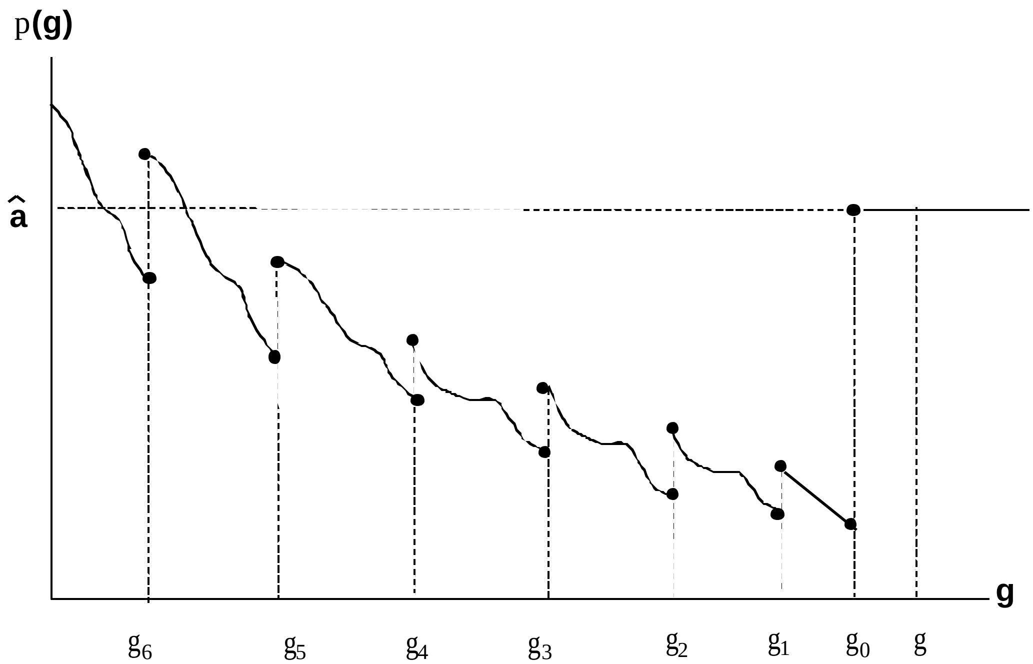

Figure 3-1: Stationary Optimal Policy

(CRRA) or constant absolute (CARA) coefficients of risk aversion. The following represent this class of functions:

$$h(a) = \ln a \tag{3.25}$$

$$h(a) = \frac{1}{\rho} a^{\rho} \tag{3.26}$$

$$h(a) = -e^{-\rho a} \tag{3.27}$$

where ρ is a constant. I have applied the characterization developed in [21] and [39] for (3.25) below, with $h(a) = \ln a$. In their papers, Hakansson and Phelps provide solutions to the other two cases, which may be readily applied.

3.3.1 Logarithmic Benefit Function

Condition 3.4

$$h(\cdot) = \ln a$$

The logarithmic benefit function has the property that the optimal split of the $(\gamma - g_0)$ is not dependent on g_0. In effect, as with other benefit functions in this class, the logarithmic function eliminates wealth effects. The function also allows us to satisfy Conditions 3.1-3.3.

Proposition 3.4 *Under Condition 3.4,*

$$\gamma_0 = \frac{(1-\delta)W\left(-e^{-\delta\alpha-\delta c\gamma-1}\right)}{\delta c} + \gamma \tag{3.28}$$

where $W(\cdot)$ is Lambert's W function (see Appendix). Moreover,

$$\gamma_T = \gamma - \frac{1-\delta^T}{(1-\delta)\,\delta^T}\left(\frac{1-\delta^{T+1}}{1-\delta^T}\right)^{\frac{1-\delta^{T+1}}{(1-\delta)\delta^T}} \exp((1-\delta)v_\gamma) \tag{3.29}$$

Proof. The Euler equation implies that $a_{t+1} = \delta a_t$. This can be combined with (3.17) to yield

$$a_{T,t}(\gamma - g) = \frac{\delta^t(1-\delta)(\gamma - g)}{1-\delta^T} \text{ for } t = 0 \text{ to } T-1 \tag{3.30}$$

Hence,

$$\lambda_{T,t} = \delta^t\frac{(1-\delta)}{1-\delta^T} \text{ for } t = 0 \text{ to } T-1 \tag{3.31}$$

In addition, $\widehat{a} = \frac{1-\delta}{\delta c}$.

In the first region, immediately to the left of γ_0, $T^* = 1$. We can solve for γ_0 as follows:

$$\begin{aligned} \text{Value from emitting } \widehat{a} &= V^1 \\ \ln(\widehat{a}) + \delta V(\gamma_0 + \widehat{a}) &= \ln(\gamma - \gamma_0) + \delta v_\gamma \\ \ln(\frac{1-\delta}{\delta c}) - \ln(\gamma - \gamma_0) &= \delta\alpha + \frac{\delta c}{1-\delta}(\frac{1-\delta}{\delta c} - \gamma + \gamma_0) \end{aligned} \tag{3.32}$$

The solution to this equation requires the use of Lambert's $W(\cdot)$ function. The solution is (3.28). We can solve for the boundaries of the regions, denoted γ_T.

Let γ_T for $T = 1, 2, 3, ...$ be the g which solves

$$\sum_{t=0}^{T-1} \delta^t \ln(\lambda_{T,t}(\gamma - g)) + \delta^T v_\gamma = \sum_{t=0}^{T} \delta^t \ln(\lambda_{T+1,t}(\gamma - g)) + \delta^{T+1} v_\gamma$$

Hence, the γ_T represent the boundaries of each region, where we are indifferent between a policy of going to γ in T periods and a policy of going to γ in $T-1$ periods. The logarithmic case allows us to provide closed form solutions for γ_T, which are given by (3.29). ∎

Comparative Statics

Because there is a closed-form solution, we can carry out some comparative statics on the impact of a change in a number of parameters. In particular, we examine how γ_T changes as $\gamma, c,$ and α changes:

$$\frac{\partial \gamma_T}{\partial \gamma} = 1 - \frac{1-\delta^T}{\delta^T} \left(\frac{1-\delta^{T+1}}{1-\delta^T}\right)^{\frac{1-\delta^{T+1}}{(1-\delta)(\delta^T)}} \frac{\partial v_\gamma}{\partial \gamma} \exp\left((1-\delta)\, v_\gamma\right) \tag{3.33}$$

$$\frac{\partial \gamma_T}{\partial c} = -\frac{1-\delta^T}{\delta^T} \left(\frac{1-\delta^{T+1}}{1-\delta^T}\right)^{\frac{1-\delta^{T+1}}{(1-\delta)(\delta^T)}} \frac{\partial v_\gamma}{\partial c} \exp\left((1-\delta)\, v_\gamma\right) \tag{3.34}$$

$$\frac{\partial \gamma_T}{\partial \alpha} = -\frac{1-\delta^T}{\delta^T} \left(\frac{1-\delta^{T+1}}{1-\delta^T}\right)^{\frac{1-\delta^{T+1}}{(1-\delta)(\delta^T)}} \frac{\partial v_\gamma}{\partial \alpha} \exp\left((1-\delta)\, v_\gamma\right) \tag{3.35}$$

Note that $\frac{\partial v_\gamma}{\partial \gamma}, \frac{\partial v_\gamma}{\partial c}$ and $\frac{\partial v_\gamma}{\partial \alpha}$ are negative. Hence

$$\begin{aligned}\frac{\partial \gamma_T}{\partial \gamma} &> 0\\ \frac{\partial \gamma_T}{\partial c} &> 0\\ \frac{\partial \gamma_T}{\partial \alpha} &> 0\end{aligned}$$

This implies that if the threshold were further away, the optimal policy would involve a higher T^* (for a given g). This is not surprising, in light of Lemma 3.6. Similarly, increases in the cost parameters are associated with higher T^* (for a given g).

Unfortunately, the expression for $\frac{\partial \gamma_T}{\partial \delta}$ is too complicated to be signed, so we are unable to carry out a comparative statics exercise with the discount factor.

Numerical Example

Below, we provide a numerical example and a graphical representation of the value function and the stationary optimal policy.

Parameter Values

δ	α	c	γ
0.9	0.05	0.02	100

Implied Constants

$\widehat{a}$	v_γ
5.56	-11.3

Values of γ_T

γ_0	γ_1	γ_2	γ_3	γ_4
99.66	98.61	97.51	96.31	94.97

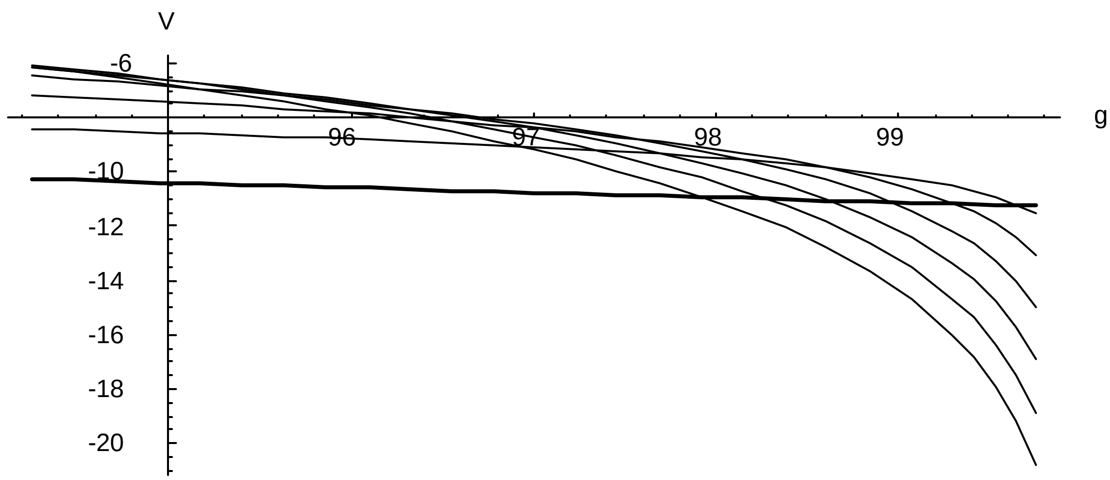

Figure 3-2: Value Function

The value function for $g \leqslant \gamma$ is the upper envelope of the lines in the figure above. In this numerical example, $\gamma = 100$. The straight line, which has a slope of $-\frac{\delta c}{1-\delta}$, represents the payoff between γ_0 and γ. The concave lines represent V^T for different values of T.

In the logarithmic case, the optimal policy within each region is linear and decreasing. The slope of each line segment decreases as g increases. The slope of the line segment is given by $-\lambda_{T,0} = -\frac{(1-\delta)}{1-\delta^T}$. Hence, in the last region before γ_0, where $T^* = 1$, the slope of the line segment is -1.

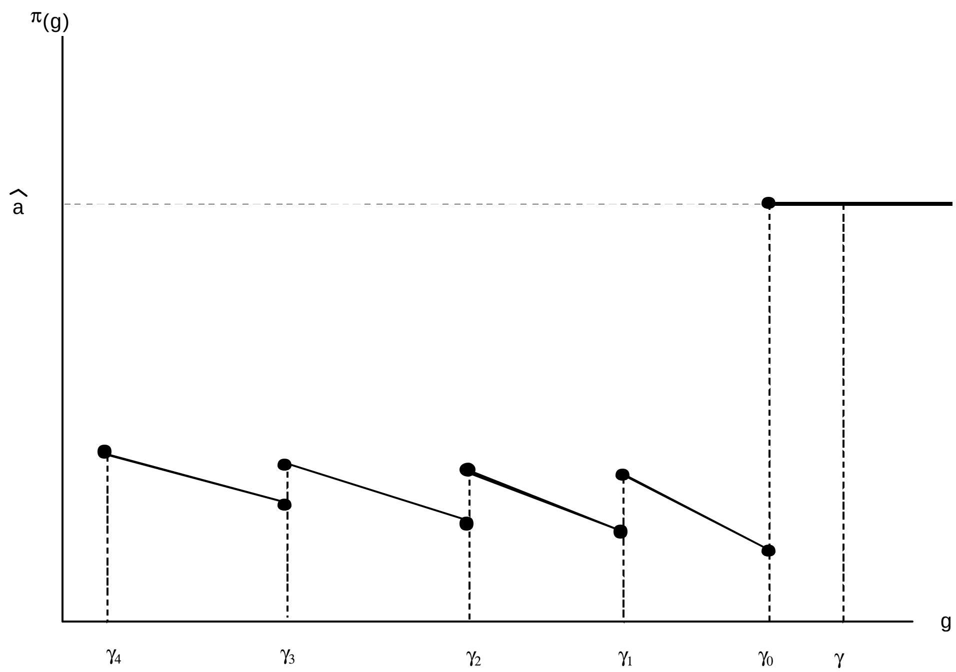

Figure 3-3: Stationary Optimal Policy with Logarithmic Benefit Function

3.4 A Computational Algorithm

The steps used to solve the threshold problems in this chapter and in Chapter 2 are analogous. They consist of the following:

1. Establish sufficient conditions such that under the optimal policy, the threshold will be reached in finite time.

2. Constrain the problem such that the threshold is reached in any given T^* number of periods and compute the optimal emission path.

3. Repeat step 2 for a range of T^*'s.

4. Select the T^* that maximizes the value function.

Following the above algorithm directly (the "brute force" method) could take a very long time to run, since we have not placed any natural limits on the range of T^*'s. However, we enhance the procedures by using coarse information about inoptimal emissions to narrow our search. Firstly, we use the Euler equation and limits on optimal period T^* emissions to narrow the range of T^*'s whose value functions must be checked. In effect, we carry out coarse, computationally cheap, checks to limit the range of T^*'s that must be subjected to finer checks. This is a form of interpolation search algorithm.

In addition, in step 2 for a given T^*, we can speed up the search for the optimal path by using information about the optimal period 0 emissions. Again, we can eliminate a range of candidate paths using computationally cheap initial checks.

We lay out procedures that utilize these two strategies below:

3.4.1 Limit the range of T^*'s to check:

We choose an initial large upper bound $T_{\max}$ on T[5]. For every $T < T_{\max}$, we compute two extremal emission paths. The first path $\overline{P(T)}$ has a T period emission equal to $\widehat{a}$; the second path $\underline{P(T)}$ has a T period emission equal to $\gamma - \gamma_0$[6]. Both paths satisfy the Euler equation for optimal adjacent period emissions: we construct these adjacent period emissions using the L operator described in Chapter 2. The first path $\overline{P(T)}$ is a path, for a given T, which chooses the largest T period emission and still satisfies the Euler condition. If, having chosen such a path, we are unable to fully emit $\gamma - g_0$, then such a T cannot be optimal in the problem without a T-constraint. Hence we discard the set of T's where the total emissions under $\overline{P(T)} < \gamma - g_0$. The second path $\underline{P(T)}$ is a path, for a given T, which chooses the smallest T period emission and still satisfies the Euler condition. If, having chosen such a path, we still emit more than $\gamma - g_0$, then such a T cannot be optimal in the problem without a T-constraint.Therefore, we also discard the set of T's where the total emissions under $\underline{P(T)} > \gamma - g_0$.

[5] In my code, I have chosen 2501, corresponding to 2501 years from the present.

[6] The second path is only computed in the case of discontinuous problems such as the ones in chapter 3 and chapter 4.

This method significantly reduces the the computation time: with $\delta = 0.96$ and $\alpha = 0.06$, we can discard T's between 1 and 130 and those above 174. Hence, we need only check T's between 131 and 173 for optimality.

3.4.2 Limit the range of candidate optimal paths to search for a given T^*:

We now have a small range of T's to check for optimality. However, for each of these T's, we still must find the optimal emission path and value function. This involves finding the emission path that satisfies the Euler equation and yet has a total emission over T periods which sums exactly to $\gamma - g_0$. We use a binary search algorithm as follows. The extremal paths computed in 3.4.1 can be utilized as upper and lower bounds to narrow the search. For each T, we use the average of the period 0 emission from $\overline{P(T)}$ and $\underline{P(T)}$ as a starting value for a candidate emission path $P_0(T)$. We use the inverse of the L operator to construct the T period emission path $P_0(T)$ which satisfies the Euler equation. If the total emissions under $P_0(T)$ exceed $\gamma - g_0$, we take an average of the period 0 emission under $P_0(T)$ and the period 0 emission under $\underline{P(T)}$, and use the inverse L operator to construct a new candidate path $P_{-1}(T)$ which will have lower total emissions than $P_0(T)$. If the total emissions under $P_0(T) < \gamma - g_0$, we take an average of the period 0 emission under $P_0(T)$ and the period 0 emission under $\overline{P(T)}$, and use the inverse L operator to construct a new candidate path $P_{+1}(T)$ which will have higher total emissions than $P_0(T)$. Although I do not prove here that this process converges, such a proof is easily constructed due to the concavity of the objective function, the compactness of the range of emissions, and the fact that this process is a contraction mapping. With a tolerance of 0.01% of the value function, this process always converged within 15 iterations for all discount factors analyzed.

3.5 Concluding Remarks

Analyzing the economics of abrupt climate change requires analyzing the impact of a discontinuity in the cost function at the threshold level of GHG. Incorporating the discontinuity complicates the search for the optimal policy and value function. The discontinuity causes a non-monotone optimal policy and emission path. I have fully characterized the structure of the policy and value functions and laid out an algorithm to solve the model in a computationally efficient manner.

In addition, this model extends the literature on exhaustible resources by incorporating an outside option into the standard problem. The presence of the outside option complicates the optimal policy, creating regions of the state variable where it is optimal to exercise the outside option at different points in the future. It endogenizes a minimum level of consumption below which it is not optimal to continue on a path of asymptotically smaller consumption.

Chapter 4

The Interaction of A Discontinuous Threshold and Discount Factor Uncertainty

The threshold models presented in the previous chapters ignore all forms of uncertainty. Clearly, there are numerous sources of uncertainty in global warming problems. Heal and Kristrom (2002) have outlined at least three general sources of uncertainty. Firstly, there is significant uncertainty regarding the magnitude of expected climate change and the parameters of the underlying climate models. Secondly, there is considerable uncertainty surrounding the economic impact of climate change. Finally, there is no consensus on the types, magnitude and impact of optimal policies to mitigate climate change. This last type of uncertainty is primarily the result of preference uncertainty: how to measure and aggregate the preferences of all concerned individuals across space and time. Heal and Kristrom point out that in cost-benefit analyses of climate change problems, the conclusions are most sensitive to two forms of preference uncertainty: the uncertainty regarding the choice of an appropriate discount factor and uncertainty regarding the index of risk aversion.

In this chapter, I examine the impact of uncertainty in preference aggregation across time. Specifically, I analyze the impact of discount factor uncertainty on the optimal policy in a particular application of the threshold model in chapter 3, that of the potential collapse of the North Atlantic thermohaline circulation (THC) process. I calibrate the model presented in the previous chapter to the specific parameters suggested by an examination of this particular global warming threshold. Accounting for discount factor uncertainty facilitates recognition of the associated real option value[1]. The option value arises because ex ante preference uncertainty and irreversibility create a potential value to delaying the time at which a central planner would optimally choose to surpass the irreversible threshold. Current stakeholders or future generations, who might choose discount factors higher than the one chosen by the modeler, would place a higher value on averting the threshold than the one computed by the model. Accounting for this potentially higher value facilitates option-adjusted pricing of the natural asset. Without an approach incorporating discount factor uncertainty, no option values of this type are recognized in the cost-benefit analysis and hence assets with very long lives and many potential users in the distant future (such as the climate system) are undervalued relative to assets with far shorter lives and potential users only in the relatively near future (such as consumption goods or capital goods). I use the calibrated threshold model to compute one measure of the magnitude of undervaluation for this option value.

The algorithm employed to solve the calibrated threshold model can be employed to examine the effect of parameter uncertainty in any other model parameter. I specifically examine the effect of uncertainty regarding the catastrophic costs associated with a collapse in the North Atlantic thermohaline circulation process. I show that for distributions calibrated from the empirical literature, the impact of discount factor uncertainty is significantly larger than the impact of catastrophic cost uncertainty. These impacts work in the same direction, and combine to further increase

[1] There is an extensive literature on the option value or "quasi-option value" associated with irreversibilities and preference uncertainty. For a list of references, see Chichilnisky, Heal and Vercelli (1998).

the option value associated with delaying the time at which a central planner would optimally choose to surpass the irreversible threshold.

In addition, a constant discount factor cost-benefit analysis leads to a second-order distortion: it distorts the value of information which would resolve catastrophic cost uncertainty. With any given discount factor, the present value of uncertainty in the catastrophic costs is virtually negligible. With any given catastrophic cost, the present value of uncertainty in the discount factor is quite significant. In the calibrated threshold model, at a constant discount rate of 4%, we ought to be willing to pay just 0.04% of our wealth to switch from a world where post-threshold catastrophic costs amount to a sure 10% of GDP to one where they amount to a sure 0%. On the other hand, using an approach incorporating a commonly cited discount factor distribution, we ought to be willing to pay 1.43% of our wealth to make the same switch, an amount that is 36 times as much as the constant discount factor case. This multiple is increasing in the variance of the discount factor distribution.

I implement a computationally efficient method that fully utilizes the results of the previous chapters to reduce the computation speed while ensuring a global optimum. Unlike numerical threshold models, we can guarantee that the solution is an ex-ante global optimum. The low computational cost of finding the optimum facilitates an examination of model risk because large numbers of iterations are feasible. The search algorithm, which involves backward induction and iterative application of the L operator, is described in chapter 3. Notes on the algorithm for the variant of the problem in this chapter, along with the Matlab code, are provided in the Appendix.

4.1 Discount Factor Uncertainty

We now turn to the impact of discount factor uncertainty in evaluating the implications of the threshold model. Clearly, the time scales in which climate change problems in general and the possibility of a THC collapse in particular must be evaluated are measured in centuries. Consequently, the choice of discount factor δ will overwhelm the effect of any other parameter choice. At a constant discount factor of 0.96, one unit of utility in 150 years is worth 0.002 units today. At a constant discount factor of 0.99, one unit of utility in 150 years is worth 0.22 units today, which is 101x its value at the lower discount factor.

There is a significant heterogeneity of views on the current and future discount rates that are appropriate for long horizon environmental cost-benefit problems. There has been an ongoing debate on the appropriate value since at least Ramsey (1928).[2] An approach often used in careful cost-benefit analysis is to equate the discount rate to the sum of a) the pure rate of time preference and b) the product of the elasticity of marginal public utility and the growth rate of wealth. For ethical reasons, term a) the pure rate of time preference, is often taken to be close to zero: Arrow et al (2004) suggest a range of 0-0.5%. Term b) is more difficult to establish because the growth rate of future wealth is quite uncertain, and in the context of long horizon environmental problems with irreversible catastrophes could be negative or only slightly positive. The analysis of Arrow et al. suggests that even in the recent past, the growth rate of a measure of wealth that includes natural capital may be insignificantly different from zero. Hence, the appropriate net discount rate may well be significantly lower than 2%, with the discount factor being correspondingly higher than 0.98.

Despite some consensus on the approach of computing an appropriate discount rate, there remains considerable disagreement on the actual rate that ought to be used, presumably because there is disagreement on the appropriate values that the parameters should take. In a survey asking 2,160 economists their best guess of the appropriate discount rate, Weitzman (2001) received a range of responses that appear to fit a gamma-distribution with a mean rate of 4% (or an approximate discount factor of 0.96) and a standard deviation of 3%.

[2] A range of different philosophical approaches are outlined in Portney and Weyant (1999).

In this paper, we follow the approach of directly incorporating uncertainty about discount factors in a very simple way, as suggested by Weitzman. In particular, we first calibrate the planner's problem from Chapter 3 (that of solving for the optimal emissions path given a constant discount factor δ). We then ask how the optimal emission path differs for different constant discount factors. This step is merely one of iterative computation and is rendered computationally feasible by the analytic solution and efficient computation algorithm developed in Chapters 2 and 3. We assume that the potential distribution of discount rates corresponding to the discount factors is a gamma distribution as estimated by Weitzman's survey. Finally, I ask how a planner, who chooses to implement a policy that is a weighted average of the optimal policies for each constant discount factor, would behave. The weights used in the weighted average are those implied by the parametrized gamma distribution. This approach is rational for a planner who does not expect to be able to influence the discount factor distribution and does not expect to learn anything about the distribution until it is too late to change his policy. In this paper, I show that a combination of discount factor uncertainty and irreversibility leads to a material option value to delaying the date at which to surpass the γ threshold. I also give an estimate of the sensitivity of this option value to a mean-preserving spread in the gamma distribution.

Newell & Pizer have examined numerically the impact of stochastic discount factors in a dynamic model of climate change without thresholds. Their approach is to take as given the current discount factor and to project a future distribution of discount factors based on a history of US interest rates. Because of the heterogeneity of views on the current discount rate, I have adopted directly the approach suggested by Weitzman, namely one where near-term discount rates are as uncertain as rates in the far future. Of course once one takes a stochastic discount factor approach, the initial discount factor has far less influence on the results than the discount factor in the distant future. The Newell & Pizer study does not incorporate thresholds. I am specifically interested here in the interaction between the stochastic discount factor and the threshold and its implications for option value. This paper bolsters the importance of discount factor uncertainty demonstrated by Newell & Pizer by showing that in the absence of a stochastic discount factor approach, threshold effects are virtually irrelevant in cost-benefit analysis.

I note that the analytical model developed in the previous two chapters is crucial to dealing properly with discount factor uncertainty. Dynamic global warming models which involve purely numerical optimization are forced to ignore the infinite horizon nature of the problem, because computers cannot solve infinite horizon problems. The justification for instituting a near-term finite horizon (200-500 years) is that at “reasonable” discount rates (often 3%), the future stops mattering after 200 years or so. An alternative justification is that it is quite speculative to make extrapolations beyond 200 years or so. This type of argument is appropriate for quite short horizons in the context of problems in finance, but it is inappropriate in an environmental problem such as this. Firstly, by applying a relatively high constant discount rate of 3%, we ensure that the distant future is irrelevant. Secondly, in the context of irreversibilities, it is more speculative to assume that the discount rates used by humans two centuries hence will be 3% than it would be to explicitly model discount factor uncertainty. To deal with lower discount rates, the finite horizon in a purely numerical model would have to expand to 2500 years or more. The reason the models are generally limited to 500 years or so is due to computational complexity. The reality is that few existing numerical models can actually be extended out to 2500 years given the computational costs. The algorithms employed here allow us to evaluate the optimal solution at annual frequency for very low discount rates at minimal computational cost (2-3 minutes per iteration on a dual-Xeon processor PC).

Discount factors close to 1 ensure that the present value of utilities in the far future are non-negligible. Consequently when the optimization routine checks candidate paths for optimality, it must check paths that have many more periods than in the case with lower discount factors. It is this requirement which slows down the search for optimal paths when the problem has a high discount factor. As the number of periods increases, the curse of dimensionality ensures that improved hardware resources foreseeable in the near future will not on their own allow us to solve the problem. The addition of a threshold further complicates the computational problem since it is no longer sufficient to examine marginal benefits and costs only. The thrust of my approach is to streamline the search for the optimal path by melding analytical solutions and bespoke numerical methods and then use the liberated computing resources to analyze parameter uncertainty and thresholds.

In a numerical model with a finite horizon, Gjerde et al. (1999) carry out an analysis of optimal climate policy under the possibility of a future general catastrophe. My results are qualitatively consistent with theirs but are conceptually and quantitatively different. Although all their main results are stated for a discount rate of 3%, Gjerde et al. carry out sensitivity analysis on the discount rate. They do not however use a distribution for this parameter. My results are consistent with their sensitivity analysis insofar as optimal policies become dramatically more conservative at lower discount rates. They use a distribution for catastrophic damage and so are able to report an estimate of the value of the resolution of catastrophic damage uncertainty but they do not report a corresponding value for the resolution of discount factor uncertainty because they do not use a discount factor distribution. Because this paper treats catastrophic damage uncertainty and discount factor uncertainty in a similar way, my study demonstrates that the expected value of the resolution of discount factor uncertainty is far higher than the expected value of the resolution of catastrophic damage uncertainty. Gjerde et al. also restrict their planning horizon to 240 years. It is possible that as a result, the negative impact of a catastrophe even with a low discount rate, is understated.

Keller, Bolker and Bradford (2004) use a numerical model of an uncertain climate threshold that is also calibrated to a THC collapse. However, they use a fixed discount rate of 3%. Their model is notable because their time profile of optimal emissions, computed numerically, is the same as the one derived analytically in Chapter 3[3].

4.2 Thermohaline Circulation Collapse

Climate scientists are concerned about the possibility of abrupt climate change induced by a collapse in the North Atlantic thermohaline circulation. A simplified description of the basic ocean-atmosphere dynamics are as follows. There exists a global ocean circulation system, often called the Ocean Conveyor, which transports equatorial heat towards the poles via warm ocean surface currents. Near the poles, these currents lose heat to the cold atmosphere, becoming saltier and denser due to evaporation, thereby sinking to the deep ocean and creating a void that continues to draw more warm surface waters towards the poles. This process is known as the "thermohaline circulation" because it is driven by changes in temperature and salt content. It significantly reduces equator-to-pole temperature differences throughout the year and its effect is more pronounced in the winter months when the ocean-atmosphere temperature gradient is higher. The Gulf Stream, a branch of this global thermohaline circulation process, is responsible for warming the climate of the eastern United States and western Europe by as much as 5 degrees Celsius. See Gagosian (2003) for an introductory description and NRC Committee on Abrupt Climate Change (2002) for a more detailed analysis.

The strength of the North Atlantic thermohaline circulation is reduced by net freshwater input into the system by melting glaciers and Arctic sea ice. Freshwater reduces salinity in the North Atlantic, limiting the density of the surface waters and reducing their tendency to sink to the deep ocean.[4] There is evidence that the

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[3] When an earlier version of chapter 1 was presented at the Princeton Environmental Institute in January 2001, the late David Bradford pointed out the similarity between Figure X in Chapter 1 and Figure 5 of Keller, Bolker and Bradford (2004).

[4] See Baumgartner & Reichel (1975) and Broecker (1997) for a comprehensive treatment.

freshwater content of the North Atlantic has been increasing steadily since the 1960s (see Dickson et al. 2002). Beyond a threshold level of freshwater content, the North Atlantic thermohaline circulation would likely collapse, shutting down the process which tempers climates in some of the wealthiest, most densely-populated regions of the world. There is geological evidence that this type of posited shutdown has occurred in the past, inducing mini Ice Ages in western Europe about 12,700 years and 8,200 years ago (see Rahmstorf 1995 and Stocker & Wright 1991).

At what threshold is the thermohaline circulation expected to collapse? Research on this question is far more limited[5] than on many other global warming parameters and the level of uncertainty surrounding any best guess is necessarily correspondingly higher. See Deutsch et al. (2002), Clark et al. (2002) and Rahmstorf & Zickfeld (2005) for very preliminary estimates of the uncertainty surrounding this threshold. A few studies have examined this question directly: Manabe & Stouffer (1993) and Stocker & Schmittner (1997). Manabe & Stouffer's climate model predicts that a quadrupling of the level of atmospheric CO2 concentration over 140 years with no further increase would lead to a near complete shutdown of the global THC. Their model predicts that with a doubling of CO_2 over 70 years with no further increase would lead to an eventual reduction in the intensity of the North Atlantic THC of more than half its original intensity[6]. Manabe & Stouffer point out that reasonable sensitivities to these scenarios imply that the THC could collapse at a doubling of CO_2 levels[7]. Stocker & Schmittner's study notes that the likelihood of THC collapse depends both on the magnitude and the speed of increase in atmospheric CO_2. They predict that an increase of CO_2 concentrations to 750ppmV (from 369.5 ppmV in 2000) within a 100 years leads to a permanent collapse of the THC. In this chapter,

[5] The TAR, for example, assigns a low likelihood to this and other forms of abrupt climate change and does not attempt to estimate a consensus view of this threshold parameter.

[6] Their model estimates that in the doubling CO2 scenario, the North Atlantic THC intensity, after falling initially, would begin to rise gradually after 150 years.

[7] Their predictions do not account for the potential freshening of near-surface water due to the melting of continental ice sheets, which, if incorporated, would increase the likelihood of THC collapse at lower CO_2 concentrations.

we show results for a calibration of the THC threshold to 750ppmV or 1597.5GtC, which is approximately 2x the current atmospheric CO_2 concentration.

4.3 Model Calibration

4.3.1 Discount Factor Distribution:

I calibrate two discount factor distributions. The base case distribution is generated by assuming that discount rates are characterized by a gamma distribution with a mean rate of 4% and a standard deviation of 3%. These are the parameters derived from Weitzman's survey 2,160 economists. In order to evaluate the impact of the intensity of preference uncertainty, I also calibrate an alternative distribution with the same mean of 4% and a higher standard deviation of 4%. The option value of delaying the date at which the threshold is surpassed is increasing in the standard deviation of the gamma distribution.

4.3.2 Benefit Function

I calibrate a Cobb-Douglas production function, with exogenous capital and labour inputs, and CO_2 emissions as a choice variable.

$$h(a_t) = A_0 K_0^{\lambda} L_0^{1-\beta-\lambda} a_t^{\beta} - p_0 a_t \tag{4.1}$$

where

A : total factor productivity

K : capital stock

L : population

a : carbon-energy services

p : per unit price of carbon-energy services

The total factor productivity, capital stock and population are assumed to be constant over time in the calibration, though the methods applied are easily extended to the case where these parameters are time-dependent but exogenous. The capital stock K and the price per unit of carbon-energy services is based on aggregated regional data from Nordhaus & Boyer. Population is actual 1998 data. Carbon-energy services is actual 1992 CO_2 emissions (ignoring land-use emissions). Total factor productivity is obtained by solving:

$$\begin{aligned} & A_0 K_0^{\gamma} L_0^{1-\beta-\gamma} a_0^{\beta} - p_0 a_0 - c_1 g_0 + c_1 g_{pre-industrial} \\ = \; & GDP_0 \\ = \; & \$27.2 \text{ trillion} \\ \Longrightarrow & \\ A_0 = \; & \frac{GDP_0 + p_0 a_0 + c_1 g_0 - c_1 g_{pre-industrial}}{K_0^{\gamma} L_0^{1-\beta-\gamma} a_0^{\beta}} \end{aligned} \tag{4.2}$$

The parameter λ, the elasticity of output with respect to capital, is assumed to be 0.30, as estimated in Nordhaus & Boyer. The elasticity of output with respect to carbon-energy services is assumed to be 0.071. This value is the real GDP-weighted average of the parameter estimates from eight regions, as estimated in Nordhaus & Boyer.

The price of carbon-energy services is also a regional aggregate derived from Nordhaus & Boyer. The extraction price of carbon energy is estimated at \$113 per ton in constant 1990\$. The weighted average mark-up, which reflects transportation costs to different energy consuming regions, is assumed to be \$212.98 per ton. The weighted average price of carbon energy services is thus assumed to be $113 + 212.98 = 325.98$ per ton.

Calibrating the benefit function to an estimate of GDP implies that the Inada condition is generally not met. For parameter values considered in this paper, this violation poses no computational problems.

Region		K_0 (in 1990 $trillion)	GDP_0 (in 1990 $trillion)	β	Energy price mark-up (in 1990 $ per ton)
US	United States	$ 12.8	$ 7.1	0.091	$ 300.00
OHI	Other High Income	$ 9.0	$ 4.7	0.059	$ 350.00
EUROPE	Europe	$ 15.0	$ 8.3	0.057	$ 400.00
EE	Eastern Europe	$ 1.2	$ 1.1	0.080	$ (38.12)
MI	Middle Income	$ 3.0	$ 1.8	0.087	$ 250.00
LMI	Lower Middle Income	$ 1.9	$ 2.1	0.053	$ (2.63)
CHINA	China	$ 0.9	$ 0.9	0.096	$ (41.09)
LI	Low Income	$ 1.6	$ 1.4	0.074	$ 18.78
Total/Average		$ 45.4	$ 27.2	0.071	$ 212.98

Source: Nordhaus & Boyer.

Figure 4-1: Benefit Function Parameters

4.3.3 Equation of Motion:

We have assumed that $\sigma = 1$. In other words, the equation of motion is assumed to be:

$$g_t = g_{t-1} + a_t \tag{4.3}$$

This weakens the magnitude of the results of this paper, though not their direction. The analysis in this paper depends on irreversibility. It is true that $\sigma < 1$ because some GHGs are annually absorbed by the atmosphere. However the broad conclusions of this paper are robust so long as σ is sufficiently close to 1 that emitting $(1-\sigma)g_{t-1}$, thereby stabilizing g, is always a sub-optimal strategy.

4.3.4 Cost Function:

The cost function we calibrate is piecewise linear with a discontinuity at the threshold γ.[8]

$$c[g_t] = \begin{cases} c_1 g_t & \text{if } g_t \leqslant \gamma \\ \alpha + c_2 g_t & \text{if } g_t > \gamma \end{cases} \tag{4.4}$$
$$\alpha, c_1, c_2 > 0; c_1 < c_2$$

CO_2 concentration, or g, is measured in GtC. Pre-industrial concentration is 583GtC (corresponds to 285ppmV). 1990 concentrations are 735GtC (corresponds to 353ppmV).

Calibration of Marginal Cost:

We calibrate c_1, the marginal cost of emissions before a THC collapse by reviewing the empirical work on damage estimates. Cline, Fankhauser and Tol estimate that the economic impact of a 2.5 degree Celsius warming by 2100 would lead to an annual cost of ranging from 1% to 1.5% of US GDP. The 2.5 degree warming corresponds to a doubling of CO_2 concentrations from their current atmospheric levels (2xCO_2). None of these estimates envisage any form of catastrophic damage caused by a collapse in the THC or by any other catastrophic occurrence. Nordhaus & Boyer estimate the breakdown of damage into catastrophic and non-catastrophic impact. They estimate that the economic impact of a 2.5 degree Celsius warming by 2100 would lead to an annual cost of 1.5% of GDP. They estimate that of this 1.5%, 0.5% relates to non-catastrophic impact and 1% relates to catastrophic impact. Nordhaus & Boyer's estimate of 0.5% for non-catastrophic damages is eggregiously low, given that Cline, Fankhauser and Tol estimate such damage to be 1.1%-1.5% for US GDP and given the consensus that the proportion of damage to other countries will likely be higher. The

[8] The analysis does not change qualitatively if $c[g_t]$ were to change from 0 to $\alpha + cg_t$ at γ rather than beyond γ.

reason for Nordhaus & Boyer's low estimate is that they employ a willingness-to-pay method of valuing economic damage. In other words, their estimate of 0.5% is an estimate of how much the world might be willing to give up today to avoid the damage from a 2.5 degree warming. The willingness to pay approach has embedded in it various assumptions about discount factors which are not necessarily consistent with the distribution of discount factors employed in this calibration[9]. Indeed, it is the argument of this paper that once discount factor uncertainty is recognized, willingness to pay increases dramatically. For our model, we require an estimate of damages conditional on 2.5 degree warming, which is necessarily higher than a willingness-to-pay estimate. We would use the average of estimates by Cline, Fankhauser and Tol, except for Nordhaus & Boyer's point that these earlier studies may overestimate damages because they did not include adaptation. We therefore calibrate c_1 such that at γ, the damage in 1990$ amounts to 1.0% of base real GDP. We assume that the damage at the pre-industrial concentration is 0.

$$.01 \times GDP_0 = c_1 \times (\gamma - g_{pre-industrial}) \tag{4.5}$$

We further assume that $c_2 = 1.5c_1$. There is no empirical work on the likely marginal cost of emissions after a catastrophe such as a THC collapse, although it is reasonable to assume that $c_2 > c_1$, given the convex cost functions assumed in integrated assessment models[10].

Calibration of Threshold and Catastrophic Cost:

We calibrate γ, the THC threshold to 750ppmV or 1597.5GtC, which is approximately 2x the current atmospheric CO_2.

[9] For example, in order to compute the willingness to pay to avoid the damage from a sea-level rise, Nordhaus & Boyer assume a constant 3% discount rate (pg. 76).

[10] Sensitivities were computed with $c_2 = 1.25c_1$ and $c_2 = 1.75c_1$. The order of magnitude of these sensitivities were so small relative to discount factor uncertainty that they further bolster the importance of discount factor uncertainty.

The calibration of α is based on estimates from Nordhaus & Boyer and Rahmstorf & Zickfeld (2005). Based on an early expert survey[11] and subsequent climate science Nordhaus & Boyer estimate that the probability of a catastrophe that leads on average to a 30% annual GDP loss, conditional on a 2.5 degree warming in 100 years is 1.2%. They estimate that the probability of such an event upon a 6 degree warming in 175 years is 6.8%. In a recent review essay that is focused solely on the THC collapse, Rahmstorf & Zickfeld estimate somewhat higher probabilities. They do not attempt to assign a GDP cost to such an eventuality. They estimate that the probability of a THC collapse, conditional on a 2.4 degree warming is 5.0%. Assuming that the annual GDP cost of a THC collapse is 30%, and a probability of a collapse at $\gamma = 1597.5$ is 5%, an expected value maximizer (or a risk-neutral planner) would assume an $\alpha = 1.5\%$ of GDP. However, expected value maximization is clearly inappropriate in this low risk-high impact context. We have used estimates of α ranging from 0% to 10% of GDP.

I compare the marginal cost estimates used here with the current price of carbon emission permits on the European Energy Exchange: the current Exchange price is approximately \$15 per ton of CO_2, or \$4 per ton of carbon. significantly higher than would be implied by c_1 alone. Based on the calibration above, the marginal cost would be \$0.27 per ton of carbon. Note that in the recent past, European prices were closer to \$2 per ton and current US prices are about \$0.50 per ton. Given the short history of these markets and the lack of market depth, it would be premature to draw any conclusions from these prices.

I note that an analogue of Condition 3 in Chapter 2 is satisfied: at γ, the GDP from zero emissions is zero while the GDP from emitting $\widehat{a}$ and incurring the marginal and catastrophic costs of emissions is positive.

[11] Nordhaus (1994).

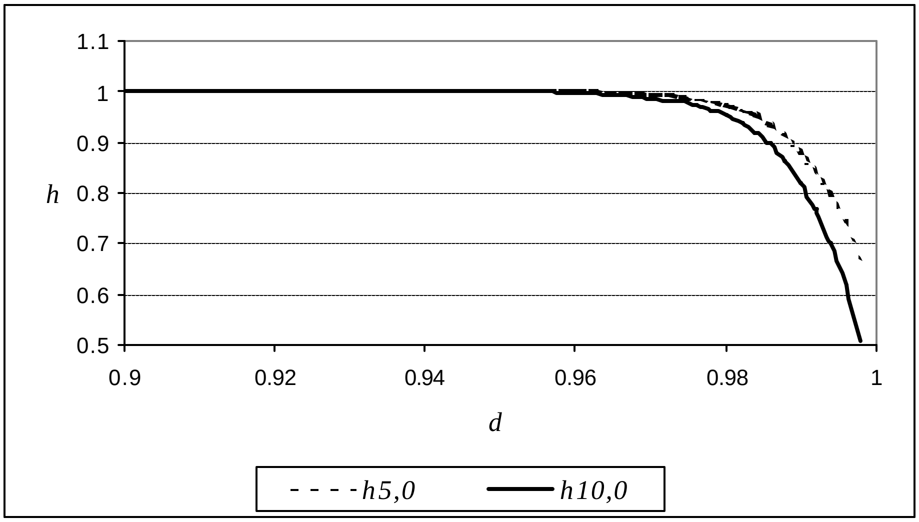

Figure 4-2: Ratio of Initial Period Optimal Emissions

4.4 Results

4.4.1 Impact of α :

We measure the importance of catastrophic costs to the optimal emission decision by examining $\eta_{5,0}$, the ratio of the initial period optimal emission given that $\alpha = 5\%$ to the initial period optimal emission given that $\alpha = 0\%$.

$$\eta_{5,0} \equiv \frac{a_0^*(\cdot) \mid \alpha = 5\%}{a_0^*(\cdot) \mid \alpha = 0\%} \tag{4.6}$$

At the constant average discount rate of 4%, the impact of a positive α on the optimal policy is minimal, i.e. $\eta_{5,0}$ is very close to 1. In other words, future catastrophic costs are irrelevant to the current optimal emission choice if we model the problem assuming a certain discount factor. If we assume a constant discount rate of 4%, $\eta_{5,0}$ is 99.9%. If we assume that $\alpha = 10\%$, the optimal initial emission for the same constant discount rate is 99.7% of the optimal initial emission with $\alpha = 0$. Below we plot the ratios $\eta_{5,0}$ and $\eta_{10,0}$ for values of δ between 0.9 and 0.998.

Clearly, with constant discount rates of $2\%-4\%$, catastrophic damages amounting to $5\% - 10\%$ of GDP are virtually irrelevant. With a constant discount rate method at these rates, there appears to be little point in resolving uncertainty about economic damage estimates or about climate parameters. However, if there is some probability that discount rates ought to be below 2%, then catastrophic costs significantly reduce the optimal initial emission. Given the lack of current consensus on the appropriate discount rate and the inability to determine rates that would be chosen by future generations, any cost-benefit analysis that does not explicitly incorporate discount factor uncertainty is not robust.

The average of the optimal initial emissions over the default gamma distribution of potential discount rates is materially lower: with $\alpha = 5\%$, the average optimal initial emission is 95.1% of the initial optimal emission with the mean discount rate. With $\alpha = 10\%$, the average optimal initial emission is 93.7% of the initial optimal emission with the mean discount rate. Note that these reductions in initial annual emissions propagate to ensuing years through the Euler equation condition which ensures that a path with lower initial emissions also has a steeper decline in emissions[12]. These reductions are quite large in the context of the Kyoto Protocol which requires cuts of just 5% by 2012 for developed countries[13]. The reductions recommended by the threshold model are additive to the emission cuts envisaged in the Kyoto Protocol (which do not account for catastrophic costs).

The impact of catastrophic costs and stochastic discount factors is seen in T^*, the optimal date at which we surpass γ. $T^* = 131$ years with $\delta = 0.96$ and $\alpha = 0$, while the average T^* over the distribution of deltas with $\alpha = 10\%$ is 174 years. $T^* = 157$ years with $\delta = 0.96$ and $\alpha = 10\%$. Hence, increasing the catastrophic cost from 0 to

[12] A question arises regarding what the planner should do with any potential windfalls as time passes and some periods realize a higher than average discount rate. Unless such windfalls question the assumed distribution of discount factors, the correct approach is to save such windfalls for those periods in future when the discount rate turns out to be lower than average.

[13] With the non-ratification by the US and other countries, and the absence of an emission reduction target for developing countries, actual reductions are likely to be closer to 1%.

10% while keeping delta constant at 0.96 increases T^* by 26 years. Accounting for discount factor uncertainty increases T^* by a further 17 years.

The lack of robustness of the optimal policy under a constant discount factor provides an alternate justification for the precautionary principle. The precautionary principle is usually taken to mean that lack of scientific uncertainty should not preclude taking actions at reasonable cost to prevent irreversible environmental damage. This is the interpretation cited by Gollier, Jullien & Treich (2000). Discount factor uncertainty is a form of scientific uncertainty: it is a recognition that we do not know what prices ought to be used to translate consumption today into consumption in 200 years. Recognizing that uncertainty would have the planner take an average of optimal policies across the distribution of discount rates which is necessarily more conservative than taking an optimal policy based on a constant average discount factor.

4.4.2 Option Value recognized by incorporating discount factor uncertainty:

We measure the option value recognized by incorporating discount factor uncertainty as the ratio ρ of the risk-neutral value of the problem and the value of the problem assuming a constant discount factor, indexed by a particular catastrophic cost α:

$$\rho_z \equiv \frac{\frac{1}{n}\sum_{i=1}^{n} V(a^*(\delta_n)) \mid \alpha = z}{V(a^*(\frac{1}{n}\sum_{i=1}^{n}\delta_n)) \mid \alpha = z} \tag{4.7}$$

The denominator in (4.7) represents the present value of all future GDP, discounted at the constant average discount rate. The average is taken over a discrete approximation of the gamma distribution of discount factors, with $n = 101$. We call the denominator *global wealth at the constant discount factor*. The numerator represents

α	Global wealth at constant discount factor (in $trillions)	%age difference relative to zero catastrophic cost	Global wealth at stochastic discount factor (in $trillions)	%age difference relative to zero catastrophic cost	ρ
0%	$652.70	0.00%	$1,269.55	0.00%	1.95
1%	$652.66	0.00%	$1,266.86	-0.21%	1.94
2%	$652.63	-0.01%	$1,264.37	-0.41%	1.94
3%	$652.60	-0.01%	$1,262.11	-0.59%	1.93
4%	$652.57	-0.02%	$1,260.04	-0.75%	1.93
5%	$652.54	-0.02%	$1,258.18	-0.90%	1.93
6%	$652.52	-0.03%	$1,256.50	-1.03%	1.93
7%	$652.50	-0.03%	$1,255.00	-1.15%	1.92
8%	$652.48	-0.03%	$1,253.66	-1.25%	1.92
9%	$652.47	-0.03%	$1,252.47	-1.35%	1.92
10%	$652.46	-0.04%	$1,251.42	-1.43%	1.92
Average	$652.56		$1,259.10		1.93

Figure 4-3: Option Value of δ-uncertainty (standard deviation of discount rate = 3%)

expected present value of all future GDP, before we know the actual discount factor that will prevail. We call the numerator *global wealth at the stochastic discount factor*. The value ρ represents the degree to which we undervalue our resource wealth by supplanting the stochastic discount factor with a constant discount factor. Using Weitzman's gamma distribution, we list below the values of global wealth at the constant and stochastic discount factors for different values of the catastrophic cost α. We list also the value of ρ.

Assuming that the catastrophic cost is a certain 5% and using a constant discount factor of 0.96, our estimate of global wealth would be $652.54 trillion. Using a stochastic discount factor approach, and retaining a certain catastrophic cost, the estimate of global wealth becomes $1258.18 trillion. The difference between these two numbers, $605.64 trillion, represents the expected present value of resolving discount factor uncertainty. If we had used a constant discount factor approach and instead recognized uncertainty in the catastrophic cost, we would value global wealth at \$652.56 trillion. The miniscule difference between \$652.56 and \$652.54 represents the expected present value of resolving catastrophic cost uncertainty at the constant discount factor. Even at the stochastic discount factor, the expected present value of resolving catastrophic cost uncertainty is small: $1259.10 - 1258.18 = \$0.92$ trillion. Clearly, the impact of discount factor uncertainty will be far greater than the impact of catastrophic cost uncertainty.

Evaluating global wealth at the constant discount factor ensures that little or no value is assigned to the resource $(\gamma - g)$ in those states where the discount factor is close to 1. These are precisely the states where the resource is most valuable. As a result, we are led to the following incredible conclusion: that we ought to be willing to give up just 0.04% of our wealth in order to switch from a world where $\alpha = 10\%$ for sure to one where $\alpha = 0\%$ for sure. The stochastic discount factor valuation on the other hand states that we ought to be willing to pay up to 1.43% of our wealth to to switch from a world where $\alpha = 10\%$ to one where $\alpha = 0\%$. This fraction of our wealth is 36 times as much as that in the constant discount factor case.

The source of this difference lies in the valuation of the perpetuity with cash flow $-\alpha$ at time T^*. The value of the perpetuity under the mean constant discount factor is smaller in magnitude than the average of values under the distribution of discount factors. Hence, under constant discounting, the difference in value between a world with $\alpha = 10\%$ and one where $\alpha = 0\%$ is orders of magnitude smaller than under stochastic discounting.

I note that for every discount factor, the cash flows being considered are always the same after T^*. The difference in impact for different discount factors lies in the present valuation of such cash flows. In a different type of problem, it is routine to pre-multiply the values from a problem by the factor $(1-\delta)$. Such pre-multiplication ensures that cash flows which are the same in future value are evaluated equally in present value terms. I do not employ such pre-multiplication here because the point is to show that a different discount factor leads to a different present value for the same cash flow and changes our perspective on current trade-offs. Even though the cash flows are the same, the average stochastic discount factor places a different present value on those cash flows than the constant average discount factor. An analogy can be drawn to the valuation of a bond: the nominal cash flows are always fixed, but a stochastic discount factor valuation approach values the bond differently because one must buy the basket of cash flows by paying in current dollars. Similarly, if we have to make sacrifices of wealth denominated in present value to delay abrupt climate change, then the stochastic discount factor approach gives some probability weight to those possibilities where the trade-off is dear. This approach treats present value as the numeraire for all wealth, and accounts for the possibility that the relative prices between the numeraire and future values are not constant.

Because this calibration assumes a fixed population and capital stock among other assumptions, the absolute dollar values are not meaningful. However, the ratios and percentages are meaningful. A conclusion of this analysis is that if we take discount factor uncertainty seriously, then the world ought to be willing to expend a far greater proportion of current wealth than is suggested by constant discount factor cost-benefit analysis in order to change the state of the climate from one in which catastrophic costs are a sure 5% of GDP to one in which they are a sure 0%.

To put this in perspective, one estimate of the value of global wealth can be computed by using the annual GDP/wealth ratio assumed by Arrow et al (2005). They use a value of 0.15 for poor and oil-rich countries and a value of 0.20 for industrialized countries. Using this assumption, global wealth measured without stochastic discount factors would amount to $148 trillion in 1990 dollars. Assuming a $\rho = 1.93$, global wealth measured with stochastic discount factors amounts to $286 trillion in 1990 dollars and $393 trillion in 2005 dollars. 0.90% of this amount is $3.5 trillion in 2005 dollars. This large number is equivalent to 22% of the market value of the Dow Jones Wilshire Broad Market Index, a broad measure of US equity market value. If we were to use a constant average discount factor approach, the corresponding

a	Global wealth at stochastic discount factor (in $trillions)	%age difference relative to zero catastrophic cost	*r*
0%	$1,989.64	0.00%	3.05
1%	$1,982.31	-0.37%	3.04
2%	$1,975.47	-0.71%	3.03
3%	$1,969.15	-1.03%	3.02
4%	$1,963.33	-1.32%	3.01
5%	$1,958.00	-1.59%	3.00
6%	$1,953.15	-1.83%	2.99
7%	$1,948.76	-2.05%	2.99
8%	$1,944.81	-2.25%	2.98
9%	$1,941.28	-2.43%	2.98
10%	$1,938.13	-2.59%	2.97
Average	$1,960.37		3.00

Figure 4-4: Option Value of δ-uncertainty (standard deviation of discount rate = 4%)

number would be $100 billion which is a relative pittance.

4.4.3 Option Value sensitivity:

Option values are driven by the variance in the underlying random variable, which in this case is the stochastic discount factor. I compute ρ for the alternate distribution of the discount rate whose mean is 4% and whose standard deviation is 4% (instead of the base case distribution with mean of 4% and standard deviation of 3%).

The alternate distribution leads to significantly higher values of ρ and implies that we ought to be willing to pay up to 2.59% of our wealth to switch from a world where $\alpha = 10\%$ to one where $\alpha = 0\%$.

Taking into account discount factor uncertainty reduces the optimal initial emission (and the subsequent future path of emissions). It also leads to a delay in T^*. With a higher mean-preserving spread in the gamma distribution of discount rates, the optimal initial emissions fall further and T^* is further delayed. With a gamma distribution with a standard deviation of 4% and $\alpha = 10\%$, the average initial optimal emission is just 89% of the case where δ is fixed at 0.96 and $\alpha = 0$. The average T^* increases to 214 years from a fixed 131 years. Hence, a small increase in discount factor uncertainty significantly leverages the impact of catastrophic costs on initial emissions.

4.5 Conclusions

I have shown that accounting for discount factor uncertainty makes catastrophic costs enormously important. Conversely, cost-benefit analysis conducted without incorporating discount factor uncertainty renders catastrophic costs and catastrophic cost uncertainty virtually irrelevant. Optimal policies and wealth measurement computed without incorporating discount factor uncertainty are not robust to the possibility that the appropriate discount factor may turn out be higher than the assumed value. This is the first contribution to compare the impact of uncertainty in the catastrophic cost versus the impact of uncertainty in the valuation of that cost and to show that the latter is more important than the former.

In future research I hope to compare the magnitudes of the positive option value embedded in a long-horizon natural asset with the negative option value embedded in short-horizon emission costs. It has been argued that because these two option values work in opposite directions, the net impact of accounting for optionality in environmental policy is unclear[14]. A proper accounting of horizons and underlying variance in these two options is necessary to sign the net impact. I believe the hybrid analytical-computational approach used in this paper can facilitate an examination of this question in an infinite horizon framework with multiple sources of uncertainty.

[14] See, for example, Pindyck (2000).

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0.1 Appendix Proofs

Lemma 0.2 (3.2) *For all t, $\lambda_{T+1,t} < \lambda_{T,t}$. In addition, $V^{T\prime}[g_0] > V^{T+1\prime}[g_0]$.*

Proof. *Step 1*: Suppose, in contradiction to the first statement, $\lambda_{T+1,t} \geq \lambda_{T,t}$ for some t. Since $\sum_{t=0}^{T-1} \lambda_{T,t} = \sum_{t=0}^{T} \lambda_{T+1,t} = 1$, there must a τ s.t. $\lambda_{T+1,\tau} \leqslant \lambda_{T,\tau}$. From the Euler equation, we have $h'(a^*_{T,t}) = \delta^{\tau-t} h'(a^*_{T,\tau})$ and $h'(a^*_{T+1,t}) = \delta^{\tau-t} h'(a^*_{T+1,\tau})$. The concavity of h implies that $h'(a^*_{T+1,t}) \leqslant h'(a^*_{T,t})$ and $h'(a^*_{T+1,\tau}) \geq h'(a^*_{T,\tau})$. Hence, we have

$$h'(a^*_{T+1,t}) \leqslant h'(a^*_{T,t}) = \delta^{\tau-t} h'(a^*_{T,\tau}) \leqslant \delta^{\tau-t} h'(a^*_{T+1,\tau}) = h'(a^*_{T+1,t})$$

The only way this can be true is if $\lambda_{T+1,\tau} = \lambda_{T,\tau}$ and there is no τ s.t. $\lambda_{T+1,\tau} < \lambda_{T,\tau}$. But then, since $\sum_{t=0}^{T-1} \lambda_{T,t} = \sum_{t=0}^{T} \lambda_{T+1,t}$, we must have a τ s.t. $\lambda_{T+1,\tau} = 0$, which is clearly not optimal by Condition 2.2.

Step 2: For any fixed T, we can differentiate (3.18) to yield

$$V^{T\prime}[g_0] = -\sum_{t=0}^{T-1} \lambda_{T,t}\delta^t h'[\lambda_{T,t}(\gamma - g_0)]$$

Since $h'(\lambda_{T,t}(\gamma - g_0)) = \delta h'(\lambda_{T,t+1}(\gamma - g_0))$ from the Euler equation, we have

$$V^{T\prime}[g_0] = -\sum_{t=0}^{T-1} \lambda_{T,t} h'[\lambda_{T,0}(\gamma - g_0)] = -h'[\lambda_{T,0}(\gamma - g_0)]$$

Since $\lambda_{T+1,0} < \lambda_{T,0}$, it follows that $V^{T\prime}[g_0] > V^{T+1\prime}[g_0]$. ■

Lemma 0.5 (3.5) *The optimal policy correspondence π is stationary.*

Proof. *Step 1*: Proposition 3.2 establishes the existence of V. By Bellman's Principle of Optimality, V satisfies the following equation for all $g \in \mathbb{R}_+$:

$$V[g] = \max_{a \in \mathbb{R}_+}\{h(a) - c(g) + \delta V[g + a]\}$$

We know that under the optimal policy, $a_t^* \leqslant \overline{a}$. Hence, in particular, V satisfies the following equation for all $g \in S$:

$$V[g] = \max_{a \in A}\{h(a) - c(g) + \delta V[g + a]\}$$

Define $G^* : S \to A$ by

$$G^*[g] = \arg\max_{a \in A} V[g]$$

Define the correspondence $\Gamma : S \to \mathbb{R}_+$ by

$$\Gamma(g) = [0, \overline{a}] \forall g \in S$$

Step 2: Fix g, and let $\{g_n\}$ be any sequence in S that converges to g. Choose $a_n \in G^*(g_n), \forall n$. Since $\Gamma(\cdot)$ is continuous, $\exists$ a sequence $\{a_n\} \to a \in \Gamma(g)$. Let $z \in \Gamma(g)$. Since $\Gamma(\cdot)$ is continuous, $\exists$ a sequence $\{z_n\} \to z$, with $z_n \in \Gamma(g_n), \forall n$. Since $V[g_n, a_n] \geq V[g_n, z_n], \forall n$, and V is continuous on $S \times A$, we have $V[g, a] \geq V[g, z] \forall g \in S$. Since this holds for any $z \in \Gamma(g)$, it follows that $a \in G^*(g)$. Hence, G^* is continuous.

Step 3: Since G^* is continuous, there is a function $\pi^* : S \to A$ such that for each $g \in S, \pi^*(g) \in G^*(g) \subset \Gamma(g)$. The function π^* defines a stationary optimal strategy, that by definition satisfies at all $g \in S$,

$$\begin{aligned} V(g) &= h(\pi^*(g)) - c(g) + \delta V(g + \pi^*(g)) \\ &\geq h(a) - c(g) + \delta V(g + a) \\ \text{for all } a &\in A \end{aligned}$$

Step 4: Let the optimal policy π be defined as

$$\pi(g) = \begin{cases} \pi^*(g) & \text{if } g < \gamma_0 \\ \widehat{a} & \text{if } g \geq \gamma_0 \end{cases}$$

Since π^* is stationary, π is stationary. ■

0.2 Appendix on Lambert's W function

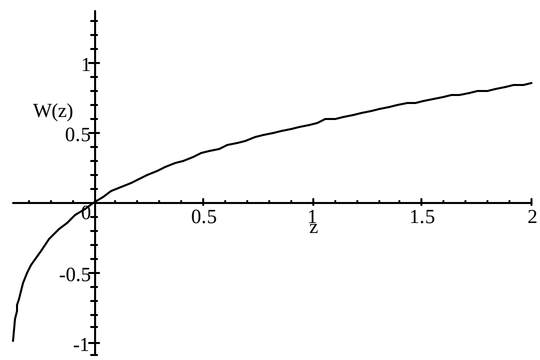
Lambert's W function, $W(z)$ is defined to be the function satisfying

$$W(z)e^{W(z)} = z \tag{8}$$

The function has a series expansion

$$W(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^{n-2}}{(n-1)!}z^n=z-z^2+\frac{3}{2}z^3-\frac{8}{3}z^4+\frac{125}{24}z^5-\frac{54}{5}z^6+... \qquad (9)$$

The principal branch of W is a single-valued function for $z\in[-\frac{1}{e},\infty)$.[15]



Lambert's W function

An application of the W function lies in the analytic solution of exponential equations. Note that by definition, the solution of $a=xe^x$ is $x=W(a)$. However, as pointed out by Lémeray in [29], and reviewed by Corless et al. in [11], a variety of other equations can be solved in terms of W. For example, the solution of $xb^x=a$ is $x=\frac{1}{\ln b}W(a\ln b)$. The result employed in (3.28) is that the solution to $ka^x=x+b$ is

$$x=-\frac{W\left(-(\ln a)\,ake^{-b}\right)}{\ln a}-b. \qquad (10)$$

[15] For $-\frac{1}{e}\leqslant z<0$, W has a subsidiary branch with $W(z)\leqslant -1$. We require only the principal branch to solve the equations in this paper.

0.3 Appendix Notes on the Algorithm and Matlab Code for Chapter 4

1. The Euler equation for optimal adjacent period emissions before T^* is:

$$h'(a^*_{t-1}) = \delta(h'(a^*_t) + c_1) \tag{11}$$

   This is the analogue of equation 3.14.

2. The benefit function is given by:

$$h(a) = A_0 K_0^{\lambda} L_0^{1-\beta-\lambda} a_t^{\beta} - p_0 a_t \tag{12}$$

   The per-period marginal benefit is given by:

$$h'(a) = \beta A_0 K_0^{\lambda} L_0^{1-\beta-\lambda} a_t^{\beta-1} - p_0$$

3. The optimal emission after γ is surpassed, $\widehat{a}$ is defined as:

$$h'(\widehat{a}) = \frac{\delta c_2}{1-\delta} \tag{13}$$

   which implies:

$$\widehat{a} = \exp\left(\frac{1}{\beta-1}\left(\ln\left(\frac{-p_0 + p_0\delta - \delta c_2}{\beta A_0 L_0 (-1+\delta)}\right) - (\ln K_0)\,\lambda + (\ln L_0)\,\beta + (\ln L_0)\,\lambda\right)\right) \tag{14}$$

   To find a, given $h'(a) = z$

4. The value function at γ is the constant:

$$V(\gamma) = v_\gamma \equiv \frac{1}{1-\delta}\left(h(\widehat{a}) - \frac{\delta c_2}{1-\delta}\widehat{a} - \delta(\alpha + c_2\gamma)\right) - c_1\gamma \tag{15}$$

5. The value of a surpassing emission of $\widehat{a}$ followed by $\widehat{a}$ forever is:

$$h(\widehat{a}) - c_1 g + \frac{\delta}{1-\delta}(h(\widehat{a}) - \frac{\delta c_2}{1-\delta}\widehat{a} - \alpha - c_2(g + \widehat{a})) \tag{16}$$

1. The value of γ_0 is defined by the following equation of indifference:

$$\begin{aligned} & h(\gamma - g) + \delta\left(h(\widehat{a}) - c_1\gamma + \frac{\delta}{1-\delta}\left(h(\widehat{a}) - \frac{\delta c_2}{1-\delta}\widehat{a} - \alpha - c_2\gamma - c_2\widehat{a}\right)\right) \\ = \; & h(\widehat{a}) + \frac{\delta}{1-\delta}\left(h(\widehat{a}) - \frac{\delta c_2}{1-\delta}\widehat{a} - \alpha - c_2 g - c_2\widehat{a}\right) \end{aligned} \tag{17}$$

2. In all other respects, the search algorithm employed in the following code is as described in Chapter 3.

Matlab Code

The Matlab code below will be available as text files at www.columbia.edu/~sgb2 or a successor webpage.

Main Code

```
% Script M-file for Threshold Model Calibration to THC collapse%
clear;
rand('state',0);
format short g;
warning off MATLAB:divideByZero;
```

```
% Set fixed parameters %
% Benefit function %
global Lambda Beta Energyprice;
Lambda = 0.30;
Beta = 0.071;
Energyprice = 325980; % 1990$ mils per GtC %
% Equation of Motion %
global Sigma;
Sigma = 1;
% Cost function %
global Gamma AlphaLow AlphaHigh C1 C2;
Gamma = 1597.5;
AlphaLow = 0.00; % Note that Alpha in the dissertation is equal to
Alpha times Gdp0 in the code
AlphaHigh = 0.1;
C1 = 268.3;
C2 = 1.5*C1; % Usually 1.5*C1.  Needs to be strictly greater than C1
% Set Initial Values %
global G0 K0 L0 A0 Gdp0 GPREIND
G0 = 781.0; % should be 781 %
GPREIND = 583.0; % should be 583.0 %
K0 = 45398000.0;
L0 = 6131.4;
A0 = 7.35; % A0 is the initial period emission, not TFP.
Gdp0 = 27214219.0;
% Find TFP by solution %
global TFP
TFP = (Gdp0 + Energyprice*A0 + C1*G0 -
```

```
 C1*GPREIND)/((K0^Lambda)*(L0^(1-Beta-Lambda))*(A0^Beta));
%
% Generate Distribution of Uncertain Parameters %
% Distribution of Alpha %
AlphaSet = [AlphaLow:0.01:AlphaHigh];
%
% Distribution of Delta %
% Gamma Distribution for interest rates %
global Nrand ScaleP ShapeP Rrand DeltaDist;
Nrand = 101;
ShapeP = 16/9; % Weitzman Alpha % % Default value should be 16/9 % %
Alternate value should be 1/1 %
ScaleP = 9/4; % Reciprocal of Weitzman Beta % % Default value should
be 9/4 % % Alternate value should be 4/1 %
Rrand = random('gam', ShapeP, ScaleP, Nrand, 1);
rand('state',0);
DeltaDist = 1 - Rrand./100;
% Discretize Delta to nearest .2
DeltaSet = min((round(DeltaDist.*500))/500,0.998);
%
% ******************************** %
% Proceed with Computation
% ******************************** %
%
% Create the Result matrix
Result = [1:13];
%
% Loop through each potential Alpha
```

```
global Alpha;
for Alpha = AlphaSet;
 %
 % Loop through each potential Delta %
 global Delta Maxt;
 for cntr_Delta = 1:Nrand;
 Delta = DeltaSet(cntr_Delta,1);
 % Compute a-hat, V-Gamma, Gamma0 %
 global TotEm Ahat Vgamma Vsurpass Gamma0
 TotEm = Gamma - G0;
 Ahat = exp((log((Delta*Energyprice-Energyprice-Delta*C2)/
(Beta*TFP*L0*(Delta-1)))-Lambda*log(K0)+Beta*log(L0)
+Lambda*log(L0))/(Beta-1));
 Vgamma = H(Ahat)/(1-Delta) - (Delta*C2*Ahat)/((1-Delta)^2)
 - (Delta*(Alpha*Gdp0+C2*Gamma))/(1-Delta) - C1*Gamma;
 Gamma0=fzero(@FindGamma0,[Gamma-Ahat Gamma]);
 % Create T-trial matrix %
 [Maxt,ETMin0,ETMax0,CMat,FinSum,Opt] = MatSize(Delta);
 T0 = (1:Maxt);
 W = abs(isinf(CMat)-1);
 CMatSum1 = ((sum(triu(CMat)))' - 1)*Delta*C1;
 CMatSum = (repmat(CMatSum1,[1 Maxt])).*W;
 % Choosing Starting Values for Em0
 Em0 = (ETMin0 + ETMax0)./2;
 HprimeEm = (repmat((Hprime(Em0))',[1 Maxt]))';
 PerPdHprime=(HprimeEm.*CMat) - CMatSum; % (HprimeEm.*CMat will have
Inf in the lower left triangle) %
 Y = (PerPdHprime ~= Inf & PerPdHprime ~= -Inf);
```

```
PerPdHprime = PerPdHprime.*Y;
PerPdEm = ((PerPdHprime + Energyprice)./
(Beta.*TFP.*(K0.^Lambda).*(L0.^(1-Beta-Lambda)))).^(1./(Beta-1));
Z = isfinite(PerPdEm);
PerPdEm(Z == 0) = 0;
SumEm = sum(PerPdEm);
E0Low = ETMin0;
E0High = ETMax0;
%
% Doing a Loop to find the Optimal Em0's
cntr = 1;
Tolerance = 0.0001;
while any(Opt.*abs(SumEm - TotEm) > Tolerance.*TotEm)
direction=SumEm>TotEm;
Em0 = Opt.*((Em0 + direction.*E0Low + (1 - direction).*E0High)./2);
E0Low = Em0 - min(E0High - Em0, Em0 - E0Low);
E0High = Em0 + min(E0High - Em0, Em0 - E0Low);
cntr = cntr + 1;
disp([cntr_Delta cntr])
HprimeEm = (repmat((Hprime(Em0))',[1 Maxt]))';
PerPdHprime=(HprimeEm.*CMat) - CMatSum;
Y = (PerPdHprime ~= Inf & PerPdHprime ~= -Inf);
PerPdHprime = PerPdHprime.*Y;
IncEmPath = (diag(PerPdHprime)' < (PerPdHprime(1,:)));
IncreasingEmPath = (repmat(IncEmPath',[1 Maxt]))';
PerPdEm = ((PerPdHprime + Energyprice)./(Beta.*TFP.*(K0.^Lambda)
.*(L0.^(1-Beta-Lambda)))).^(1./(Beta-1));
Z = isfinite(PerPdEm);
```

```
 PerPdEm(Z == 0) = 0;
 PerPdEm(IncreasingEmPath == 1) = TotEm/Maxt;
 SumEm = sum(PerPdEm);
 end
 % Generate Values %
 G=triu(triu(cumsum(PerPdEm))+G0);
 PerPdUtil = H(PerPdEm) - Cost(G);
 D = PerPdUtil.*(Delta./CMat);
 InterimValue = sum(D);
 TermValue = Vgamma*(Delta.^T0);
 Val = InterimValue + TermValue;
 [Value,TStar] = max(Val);
 PerPdEmTStar = PerPdEm(:,TStar);
 Result = [Result;[cntr_Delta Delta Alpha ShapeP ScaleP TStar Value
 Ahat Vgamma TotEm Maxt cntr PerPdEmTStar(1)]]
 end
end
save output Result;
quit
```

Functions

Cost.m

```
function y = Cost(x)
global C1 C2 Alpha Gdp0 Gamma
if x > Gamma
 y = Alpha*Gdp0 + C2.*x;
else
 y = C1.*x;
end
```

DiscountFactor.m

```
function y = DiscountFactor(x)
global Delta
y = Delta.^x;
```

FindGamma0.m

```
function y = FindGamma0(x)
% This function finds the g s.t. VSurpass(g) =
Payoff to emitting Gamma - g followed by Ahat forever
global Gamma Delta Vgamma C1
y = H(max(Gamma-x,0)) - C1.*x + Delta*Vgamma - Vsurpass(x);
```

H.m

```
function y = H(x)
global Beta TFP K0 Lambda L0 Energyprice C1 GPREIND
y = TFP*(K0^Lambda)*(L0^(1-Beta-Lambda))*(x.^Beta) - x.*
Energyprice + C1*GPREIND;
```

Hprime.m

```
function y = Hprime(x)
global Beta TFP K0 Lambda L0 Energyprice
y = Beta*TFP*(K0^Lambda)*(L0^(1-Beta-Lambda))*(x.^(Beta-1)) -
Energyprice;
```

Hprimehat.m

```
function y = Hprimehat(x)
global Delta C
y = Hprime(x) - Delta*C/(1-Delta);
```

MatSize.m

```
function [m,etmin,etmax,cmat,finsum,opt] = MatSize(x)
global C1 TotEm Ahat Vgamma Vsurpass Gamma Gamma0 Energyprice Beta
 TFP K0 L0 Lambda
```

```
% Create matrix for LrgMatSize %
 LrgMatSize = 2501;
 Ttrial = triu(repmat((1:LrgMatSize)',[1 LrgMatSize]));
 X = Ttrial ~= 0; % X is a matrix of 1's with lower left
triangle equal to zeros %
 B1=X.*(x.^Ttrial); % B1 is a matrix:  first row is delta,
2nd row is delta^2, etc.  except that lower left triangle is zeros %
 T0 = (1:LrgMatSize);
 T1 = T0 - 1;
 CMat=1./(B1./x); % CMat is a matrix:  first row is 1, 2nd row
 is 1/delta, 3rd row is 1/delta^2 etc.  except that lower
left triangle is infs %
 FinSum1 = (x*C1*(1-x.^T1)/(1-x))';
 FinSum = X.*(repmat(FinSum1,[1 LrgMatSize])); % finSum is a matrix:
first row is 0, 2nd row is 1 pd finite sum of delta*c1 etc.
except that lower left triangle is zeros %
%
% Checking T's where last period E = Ahat doesn't use up
TotEm for this
% Delta (x)
 ETMax = repmat((Ahat)',[1 LrgMatSize]);
 HprimeETMax = (repmat((Hprime(ETMax))',[1 LrgMatSize]))';
 PerPdHprimeETMax=(HprimeETMax./CMat)+FinSum; % Dividing by
Inf gives
 zeros in the lower left triangle %
 Y1 = PerPdHprimeETMax ~= 0; % Allows us to ignore the zeros
 in the lower left triangle %
 PerPdETMax = Y1.*((PerPdHprimeETMax + Energyprice)
```

```
./(Beta.*TFP.*(K0.^Lambda).*(L0.^(1-Beta-Lambda)))).^(1./(Beta-1));
 ETMaxInit = (diag(PerPdETMax))'; % Initial pd emissions for a
 T-length emission path ending with Ahat %
%
% Checking T's where last period Emission = Gamma0 uses up more than
TotEm
% for this Delta (x)
 ETMin = repmat((Gamma-Gamma0)',[1 LrgMatSize]);
 HprimeETMin = (repmat((Hprime(ETMin))',[1 LrgMatSize]))';
 PerPdHprimeETMin=(HprimeETMin./CMat)+FinSum; % Dividing by
Inf gives zeros in the lower left triangle %
 Y1 = PerPdHprimeETMin ~= 0; % Allows us to ignore the zeros in
the lower left triangle %
 PerPdETMin = Y1.*((PerPdHprimeETMin + Energyprice)
./(Beta.*TFP.*(K0.^Lambda).*(L0.^(1-Beta-Lambda)))).^(1./(Beta-1));
 ETMinInit = (diag(PerPdETMin))'; % Initial pd emissions for a
 T-length emission path ending with Gamma minus Gamma0 %
% Labeling such T's excessive
 InOpt1=sum(PerPdETMax)<TotEm;
 InOpt2=sum(PerPdETMin)>TotEm;
 InOpt=(InOpt1==1|InOpt2==1);
 Opt=(InOpt==0);
 if sum(Opt) == 0
 error('LrgMatSize insufficiently large for this delta')
 end
 if Opt(1,LrgMatSize) == 1
 error('LrgMatSize insufficiently large for this delta')
 end
```

```
 [a,b] = max(InOpt2);
m = b;
etmin = ETMinInit(1,1:b);
etmax = ETMaxInit(1,1:b);
cmat = CMat(1:b,1:b);
finsum = FinSum(1:b,1:b);
opt = Opt(1,1:b);
```